

ANCIENT RICCI FLOW SOLUTIONS ON BUNDLES

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ABSTRACT. We generalize the circle bundle examples of ancient solutions of the Ricci flow discovered by Bakas, Kong, and Ni to a class of principal torus bundles over an arbitrary finite product of Fano Kähler-Einstein manifolds studied by Wang and Ziller in the context of Einstein geometry. As a result, continuous families of κ -collapsed and κ -noncollapsed ancient solutions of type I are obtained on circle bundles for all odd dimensions ≥ 7 . In dimension 7 such examples moreover exist on pairs of homeomorphic but not diffeomorphic manifolds. Continuous families of κ -collapsed ancient solutions of type I are also obtained on torus bundles for all dimensions ≥ 8 .

Keywords. Ricci flow, ancient solutions, torus bundles, Fano manifolds, Riemannian submersions

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1. Introduction

An ancient solution $g(t)$ of the Ricci flow in this paper is one that exists on a half infinite interval of the type $(-\infty, T)$ for some finite time T and with $g(t)$ complete for each t . These solutions are important because they arise naturally when finite time singular solutions of the flow are blown up. Rigidity phenomena for ancient solutions have been established by L. Ni ([Ni09]), and by S. Brendle, G. Huisken, and C. Sinestrari ([BHS11]) under certain non-negativity conditions on the full curvature tensor. For dimension two, classification theorems were achieved by P. Daskalopoulos, R. Hamilton, and N. Sesum ([DHS12]) for closed surfaces and by R. Hamilton ([Ha95]), S.C. Chu ([Chu07]), and P. Daskalopoulos and N. Sesum ([DS06]) for complete non-compact surfaces.

Examples of ancient solutions in higher dimensions do exist. The first examples (in dimension 3) were due to Fateev [Fa96] and Perelman [Pe02]. Other examples in dimension 3 result from the work of X.D. Cao, J. Guckenheimer, and L. Saloff-Coste ([CS09], [CGS09]) on the backwards behavior of the Ricci flow on locally homogeneous 3-manifolds. Fateev's example was then generalized by I. Bakas, S. L. Kong, and L. Ni ([BKN12]) to odd-dimensional spheres, complex projective spaces, and the twistor spaces of compact quaternion-Kähler manifolds. By studying a system of two ODEs J. Lauret ([La13]) gave explicit examples of 2-parameter families of left-invariant ancient solutions on any compact simple Lie group except $Sp(2k+1)$. Further homogeneous examples can be found in [Bu14] and [BLS16]. Finally, we mention two inhomogeneous examples in dimension 4: a compact ancient solution due to S. Brendle and N. Kapouleas ([BK14]) which is constructed by desingularizing an

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orbifold quotient of the flat torus, and a non-compact inhomogeneous example due to R. Takahashi ([Ta14]) which has the Euclidean Schwarzschild metric as backwards limit.

In this article we shall be concerned with the construction of ancient solutions on closed manifolds which are not highly symmetric except in special cases. Specifically, we shall generalize the circle bundle examples in [BKN12, Theorem 5.1] to a class of principal torus bundles over an arbitrary finite product of Fano KE (short for Kähler-Einstein) manifolds. One may view our work here as a dynamic version of the work in [WZ90], although the aim, of course, is not to reconstruct the Einstein metrics obtained in that work via the Ricci flow.

To describe our main results, let P_Q denote a principal r -torus bundle with Euler classes Q over a product of m arbitrary KE manifolds (M_i, g_i) with positive first Chern class. As in [WZ90] the integral cohomology classes in Q are taken to be allowable rational linear combinations of the first Chern classes of M_i subject to certain non-degeneracy conditions explained in detail in §2. We consider the class of connection-type metrics on P_Q determined by an arbitrary left-invariant metric on the torus fibres, arbitrary scalings of the KE metrics g_i by constants, and the harmonic representatives of the Euler classes in Q with respect to the base metric chosen. This class of metrics is preserved by the Ricci flow, and the Ricci flow equation simplifies to a system of nonlinear ODEs, which we analyse in some detail to deduce

Main Theorem. *With assumptions as described above, on each total space P_Q*

- (a) *there is an $(m + r(r + 1)/2 - 1)$ -parameter family of ancient solutions of type I for which the time-rescaled backwards limit is a multiple of the product of the Kähler-Einstein metrics on the base $M_1 \times \cdots \times M_m$. These solutions are κ -collapsed.*
- (b) *When $r = 1$ and $m \geq 2$, there is a further $(m - 1)$ -parameter family of ancient solutions of type I that is κ -noncollapsed. Their time-rescaled backwards limit is a multiple of the Einstein metric on P_Q found in [WZ90].*

Detailed descriptions of the asymptotics and curvature properties of the above ancient solutions can be found in Theorem 4.5, Theorems 3.6–3.9, and Theorem 3.11.

We also investigate the forward limits of our ancient solutions in some special situations when $r = 1$. First, we show that among the ancient solutions in part (a) of the above theorem on a circle bundle there is a unique solution which develops a type I singularity in finite time and whose time-rescaled forward limit is a multiple of the Einstein metric on the circle bundle (see Theorem 3.18). Second, when $m = 2$ we show that the ancient solutions in part (b) of the above theorem develop a type I singularity in finite time and their time-rescaled forward limit is the product of a circle bundle over one of the base factors with a Euclidean space of the same dimension as the other base factor. Furthermore, this limit is κ -noncollapsed for some $\kappa > 0$ (see Theorem 3.22).

It should be noted that none of the above compact ancient solutions can occur as finite-time singularity models of the Ricci flow. This is because such models must, on the one hand, be κ -noncollapsed at all scales by the celebrated work of Perelman ([Pe02]), and, on the other hand, be shrinking Ricci solitons when the model is compact by a result of Z.L. Zhang ([Zh07]).

However, there appears to be some independent interest in ancient solutions of the Ricci flow beyond singularity analysis, as exemplified by the classification work of Daskalopoulos-Hamilton-Sesum [DHS12] and the work of physicists on the Renormalization Group flow [Fr85]. From this vantage point, our works shows that, at least among torus bundles, families of non-solitonic ancient solutions of the Ricci flow (collapsed or not) with positive Ricci curvature do exist in abundance. The torus bundles we consider here are moreover quite diverse in their topological properties, as was shown in [WZ90]. In particular, they include manifolds with infinitely many homotopy types as well as infinitely many homeomorphism types within a fixed integral cohomology type. Furthermore, at least in dimension 7 (and most probably for infinitely many dimensions) they include pairs of homeomorphic but not diffeomorphic manifolds [KS88] (see Remark 3.19 for further discussion).

In the context of Einstein geometry, our investigation also uncovers some interesting phenomena. As mentioned above, in the circle bundle case, there is a unique ancient solution that develops a type I finite time singularity whose time-rescaled blow-up limit is the Einstein metric constructed in [WZ90]. One may regard this solution as giving a curve *that satisfies a geometric PDE* “connecting” the product KE metric on the base (Theorem 3.9(i)), which is a manifold of one dimension less, to an Einstein metric on the circle bundle (Theorem 3.18(iii)). A similar curve is constructed in Theorem 3.22(i). Such solutions are especially intriguing in cases for which the total spaces are diffeomorphic but the Einstein metrics belong to distinct path components of the Einstein moduli space. For example, this happens already for dimension 5: $S^2 \times S^3$ is a circle bundle over $S^2 \times S^2$ in countably infinitely many ways, and the unit volume Einstein metrics have Ricci curvature converging to zero.

In analysing the Ricci flow restricted to connection-type metrics, we are dealing with an ODE system which is in general not bounded in the number of dependent variables. After a suitable change of both independent and dependent variables, this system becomes a polynomial system for which we find suitable bounded sets, monotonic quantities, and differential inequalities. These are used to show that the solutions stay in these sets, which in turn allows us to deduce that the solutions exist for all times approaching $-\infty$. The geometric properties of the solutions are then established from various estimates derived from the ODE system. In order to derive part (b) of our main result, we also need to study the eigenvalues of the linearization of our polynomial system at the fixed point corresponding to the Einstein metric mentioned in part (b) of the Main Theorem.

Finally, we want to point out an interesting correspondence between the Riemannian ancient solutions of the Ricci flow we found and certain ancient solutions as well as immortal solutions of the Ricci flow equations for appropriate pseudo-Riemannian metrics. More precisely, if we replace some of the Fano KE factors in the base of our circle bundles by KE manifolds with the same dimension but with *negative scalar curvature*, and use the ansatz (2.2) to define the pseudo-Riemannian metrics, we obtain both ancient and immortal solutions for the corresponding pseudo-Riemannian Ricci flow on the resulting bundles. For further details see Remark 3.15.

The following is an outline of the rest of this article. In §2 we derive the Ricci flow equation for metrics on torus bundles over a product of Fano KE manifolds which satisfy the ansatz (2.2). In §3 we show the existence of ancient solutions on the circle

bundles and study their geometric behavior as time $t \rightarrow -\infty$. We also show the existence of one particular ancient solution whose finite time singularity model is the Einstein metric found in [WZ90, Theorem 1.4]. In §3.5 we study in detail some ancient solutions whose singularity model at time $t = -\infty$ is this Einstein metric. In §4 we prove the general existence of ancient solutions on torus bundles and discuss their geometric behavior as time $t \rightarrow -\infty$. In the appendix we compute the eigenvalues of a matrix related to the linearization of our polynomial system at the Einstein metric.

In a follow-up article we will study ancient solutions of the Ricci flow on certain $(\mathrm{SO}(3) \times \cdots \times \mathrm{SO}(3))/\Delta\mathrm{SO}(3)$ fibre bundles over an arbitrary finite product of compact quaternionic Kähler manifolds.

2. Ricci flow equations for connection-type metrics on certain torus bundles

Let $(M_i^{n_i}, g_i)$, $i = 1, \dots, m$, be KE manifolds of complex dimension n_i with Ricci tensor $\mathrm{Rc}(g_i) = p_i g_i$ for some nonzero $p_i \in \mathbb{Z}$. Let ω_i be the Kähler form associated with metric g_i . By a suitable homothetic change of g_i , we may make the following **normalization assumption** throughout this article: the cohomology class $\frac{1}{2\pi}[\omega_i]$ is integral and represents a torsion free indivisible class in $H^2(M_i, \mathbb{Z})$.

Let $\pi : P \rightarrow M_1 \times \cdots \times M_m$ be a smooth principal r -torus bundle. If we identify the Lie algebra \mathfrak{t} of the torus T^r with \mathbb{R}^r by choosing a basis $\{e_\alpha; \alpha = 1, \dots, r\}$ for \mathfrak{t} associated with a fixed decomposition $T^r = S^1 \times \cdots \times S^1$, then we get an induced action of $S^1 \times \cdots \times S^1$ on P , and the principal torus bundles over $M_1 \times \cdots \times M_m$ are classified by r Euler classes χ_α , $\alpha = 1, \dots, r$, in $H^2(M_1 \times \cdots \times M_m; \mathbb{Z})$. Here χ_α is the Euler class of the orientable circle bundle $P/T^{r-1} \rightarrow M_1 \times \cdots \times M_m$, where $T^{r-1} \subset T^r$ is the subtorus with the i th S^1 factor deleted. Let A be an automorphism of T^r . A induces a new decomposition of $T^r = S^1 \times \cdots \times S^1$, and a new basis $\tilde{e}_\alpha = A_{\alpha\beta} e_\beta$, where matrix $A = (A_{\alpha\beta})_{r \times r}$ is an integral matrix with $\det(A) = \pm 1$. The new induced action of $S^1 \times \cdots \times S^1$ on P has associated Euler classes $\tilde{\chi}_\alpha = (A^T)_{\alpha\beta}^{-1} \chi_\beta$. In the rest of the article we will fix a decomposition $T^r = S^1 \times \cdots \times S^1$ and hence we will have a fixed basis $\{e_\alpha; \alpha = 1, \dots, r\}$ for the Lie algebra \mathfrak{t} of T^r .

We are interested in those principal torus bundles whose Euler classes χ_α , $\alpha = 1, \dots, r$, are given by $\frac{1}{2\pi} \sum_i q_{\alpha i} [\omega_i]$, where $q_{\alpha i}$ are integers. Let Q denote the $r \times m$ matrix $(q_{\alpha i})$. Note that if we change the decomposition $T^r = S^1 \times \cdots \times S^1$ by A as discussed above, then the new $\tilde{Q} = (A^T)^{-1} Q$, hence (a) the ranks of Q and \tilde{Q} are the same and (b) the numbers of columns of zeros in Q and in \tilde{Q} are the same. Also note that if we switch the order of factors in product $M_1 \times \cdots \times M_m$, the new Q matrix still satisfies the properties (a) and (b). Hence we can make the following definition. We call a principal T^r bundle P_Q **non-degenerate** if the associated matrix Q has rank r and has no column of zeros. This assumption implies $r \leq m$. When $r = m$, by applying a suitable automorphism (equivalently changing the decomposition of the torus into a product of circles), one sees that P is a finite quotient of a product of circle bundles over M_i . Hence the non-trivial cases begin with $m = 2$ for circle bundles and $r = 2, m = 3$ for torus bundles, and the corresponding minimum dimensions of P are respectively 5 and 8.

On a torus bundle P_Q , using Hodge theory, we may choose an \mathbb{R}^r -valued principal connection $\sigma = (\sigma_1, \dots, \sigma_r)$ such that the curvature form F^α of σ_α is given by

$$(2.1) \quad F^\alpha = d\sigma_\alpha = \sum_{i=1}^m q_{\alpha i} \omega_i.$$

Throughout this article we will use the **convention** $\tau = -t$ as the backwards time. Let $(h_{\alpha\beta}(\tau)) \doteq (h(e_\alpha, e_\beta))$ be a 1-parameter family of left-invariant metrics on T^r . Since T^r is commutative, these metrics are automatically also right-invariant. Using σ , we now obtain a family of Riemannian metrics on P_Q given by

$$(2.2) \quad g_{h,\vec{b}}(\tau) = \sum_{\alpha,\beta=1}^r h_{\alpha\beta}(\tau) \sigma_\alpha(\cdot) \otimes \sigma_\beta(\cdot) + \sum_{i=1}^m b_i(\tau) \pi_i^* g_i,$$

where $\pi_i = \tilde{\pi}_i \circ \pi$, $\tilde{\pi}_i : M_1 \times \dots \times M_m \rightarrow M_i$ is the standard projection, and $b_i(\tau) > 0$. We shall often abuse notation by referring to $\pi_i^* g_i$ as g_i . The above family of metrics makes π into a Riemannian submersion with totally geodesic fibres for each fixed τ .

The components of the Ricci tensor of $g_{h,\vec{b}}(\tau)$ are given by (see [WZ90, p.224] and [BKN12, §2.2])

$$(2.3a) \quad R(g_{h,\vec{b}}(\tau))_{\alpha\beta} = \sum_{\tilde{\alpha},\tilde{\beta}=1}^r \sum_{i=1}^m \frac{1}{2} n_i q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\alpha\tilde{\alpha}}(\tau) h_{\beta\tilde{\beta}}(\tau)}{b_i^2(\tau)},$$

$$(2.3b) \quad R(g_{h,\vec{b}}(\tau))_{k\alpha} = 0,$$

$$(2.3c) \quad R(g_{h,\vec{b}}(\tau))_{kl} = \text{Rc}(g_i)_{kl} \frac{1}{b_i(\tau)} - \sum_{\tilde{\alpha},\tilde{\beta}=1}^r \frac{1}{2} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} (g_i)_{kl} \frac{h_{\tilde{\alpha}\tilde{\beta}}(\tau)}{b_i^2(\tau)},$$

where in the last equation k and l are the indices used for an orthonormal frame $\{e_k^{(i)}\}$ for the factor M_i with respect to the metric $b_i(\tau) g_i$. Notice that equation (2.3b) automatically holds for all τ as a result of our choice of connection because the harmonicity of the curvature form for all values of τ is precisely the Yang-Mills condition for the connection metric $g_{h,\vec{b}}(\tau)$ (see Proposition 9.36 of [Bes87]). Hence the Ricci tensor can be written as

$$(2.4) \quad \begin{aligned} \text{Rc}(g_{h,\vec{b}}(\tau)) &= \sum_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}=1}^r \sum_{i=1}^m \frac{1}{2} n_i q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\alpha\tilde{\alpha}}(\tau) h_{\beta\tilde{\beta}}(\tau)}{b_i^2(\tau)} \sigma_\alpha \otimes \sigma_\beta \\ &\quad + \sum_{i=1}^m \left(\frac{p_i}{b_i(\tau)} - \sum_{\tilde{\alpha},\tilde{\beta}=1}^r \frac{1}{2} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\tilde{\alpha}\tilde{\beta}}(\tau)}{b_i^2(\tau)} \right) b_i(\tau) g_i. \end{aligned}$$

Since we choose $\tau = -t$, a solution to the Ricci flow for time $t \in [0, T)$ corresponds to a solution of the backwards Ricci flow for $\tau \in (-T, 0]$. With this convention, the

backwards Ricci flow of $g_{h,\vec{b}}(\tau)$ is given by the ODE system

$$(2.5a) \quad \frac{dh_{\alpha\beta}}{d\tau} = \sum_{\tilde{\alpha}, \tilde{\beta}=1}^r \sum_{i=1}^m n_i q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\alpha\tilde{\alpha}}(\tau) h_{\beta\tilde{\beta}}(\tau)}{b_i^2(\tau)}, \quad 1 \leq \alpha, \beta \leq r,$$

$$(2.5b) \quad \frac{db_i}{d\tau} = 2p_i - \sum_{\tilde{\alpha}, \tilde{\beta}=1}^r q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\tilde{\alpha}\tilde{\beta}}(\tau)}{b_i(\tau)}, \quad 1 \leq i \leq m.$$

If $g_{h,\vec{b}}(\tau)$ has long time existence, then $g_{h,\vec{b}}(\tau)$ is an ancient solution of the Ricci flow since our bundles are compact.

Remark 2.1. Note that if we require the diagonal condition $h_{\alpha\beta}(\tau) = a_\alpha(\tau)\delta_{\alpha\beta}$ after we fix an appropriate decomposition $T^r = S^1 \times \cdots \times S^1$, we get from (2.5a) that for $\alpha \neq \beta$

$$\sum_{i=1}^m n_i q_{\alpha i} q_{\beta i} \frac{a_\alpha(\tau) a_\beta(\tau)}{b_i^2(\tau)} = 0 \quad \text{for all } \tau.$$

Naively one would expect that $n_i q_{\alpha i} q_{\beta i} = 0$ for any i and $\alpha \neq \beta$. This then implies following conditions on $q_{\alpha i}$. After a permutation of the indices i , q_{11}, \dots, q_{1i_1} would be nonzero while $q_{1i} = 0$ for all $i > i_1$, and for each $\alpha = 2, \dots, r$, we can arrange for $q_{\alpha i_{\alpha-1}+1}, \dots, q_{\alpha i_\alpha}$ to be nonzero and $q_{\alpha i} = 0$ for all the other i . For each $\alpha = 1, \dots, r$ let $P_\alpha \rightarrow M_{i_{\alpha-1}+1} \times \cdots \times M_{i_\alpha}$ be the principal circle bundle defined by first Chern class $\sum_{i=i_{\alpha-1}+1}^{i_\alpha} q_{\alpha i} \omega_i$. Let $\hat{\pi}_\alpha : M_1 \times \cdots \times M_m \rightarrow M_{i_{\alpha-1}+1} \times \cdots \times M_{i_\alpha}$ be the natural projection map. Then the principal torus bundle P_Q is isomorphic to the product bundle $\hat{\pi}_1^* P_1 \times \cdots \times \hat{\pi}_r^* P_r$, and the Ricci flow on P_Q decouples to the Ricci flows on the circle bundles P_α . Recall that by our definition the circle bundle P_α is non-degenerate if each $q_{\alpha i}$ is nonzero for $i = i_{\alpha-1} + 1, \dots, i_\alpha$. This is the case for which we will prove the existence of ancient solutions in next section.

Remark 2.2. As input data for the torus bundles under consideration we can certainly take (M_i, g_i) to be compact homogeneous KE spaces. These are precisely the coadjoint orbits of the compact semisimple Lie groups with the induced metric, and the resulting torus bundles will actually be homogeneous (see Proposition 3.1 in [WZ90]). The resulting Ricci flows are therefore homogeneous, as the Ricci flow preserves isometries. For this special case our analysis partially overlaps with that done in [Bu14], [Buz2], [Bo15], and [BLS16].

On the other hand, this is not the generic situation and there are many concrete examples of Fano KE manifolds whose isometry groups are far from being transitive. An interesting family of cohomogeneity 3 are the small deformations of the Mukai-Umemura 3-fold [Do08]. Examples with at most a finite automorphism group include \mathbb{CP}^2 with k generic points blown up ($4 \leq k \leq 8$). These surfaces even have a positive-dimensional moduli space of complex structures, so that the KE metrics come in continuous families. See Remark 3.12 for more comments on the isometry groups of our ancient flow solutions.

3. Ancient solutions of Ricci flow on circle bundles

In this section we consider the $r = 1$ case of the backwards Ricci flow (2.5a) and (2.5b) and study the existence and asymptotic geometric properties of the ancient solutions.

Using notations $h_{11}(\tau) \doteq a(\tau)$ and $q_{1i} \doteq q_i$, the family of Riemannian metrics $g_{h,\vec{b}}(\tau)$ on the circle bundles P_Q becomes

$$(3.1) \quad g_{a,\vec{b}}(\tau) = a(\tau) \sigma(\cdot) \otimes \sigma(\cdot) + \sum_{i=1}^m b_i(\tau) g_i.$$

The backwards Ricci flow equation then simplifies to the system

$$(3.2a) \quad \frac{da}{d\tau} = \sum_{i=1}^m n_i q_i^2 \frac{a^2(\tau)}{b_i^2(\tau)},$$

$$(3.2b) \quad \frac{db_i}{d\tau} = 2p_i - q_i^2 \frac{a(\tau)}{b_i(\tau)}, \quad 1 \leq i \leq m.$$

Unless otherwise stated, in this section we shall assume the non-degeneracy condition that q_1, \dots, q_m are nonzero and the Fano condition that p_1, \dots, p_m are positive.

3.1. A polynomial system, its linearization and monotonicity property. To analyze system (3.2a) and (3.2b), we introduce new dependent variables

$$(3.3) \quad Y_i \doteq \frac{a}{b_i}, \quad 1 \leq i \leq m,$$

and let $Y = (Y_1, \dots, Y_m)$ be the corresponding vector in \mathbb{R}^m . We also introduce a new independent variable u by the relation

$$(3.4) \quad u = u(\tau) \doteq \int_0^\tau \frac{1}{a(\zeta)} d\zeta.$$

Since we shall need $a(\tau)$ to be positive, u increases with τ . We define the following useful functions

$$(3.5a) \quad E(Y) \doteq \sum_{i=1}^m n_i q_i^2 Y_i^2,$$

$$(3.5b) \quad F_i(Y) \doteq 2p_i Y_i - q_i^2 Y_i^2 - E(Y), \quad 1 \leq i \leq m.$$

One may now express the system (3.2a) and (3.2b) in terms of the new variables:

$$(3.6a) \quad \frac{d \ln a}{du} = E(Y),$$

$$(3.6b) \quad \frac{dY_i}{du} = -Y_i F_i(Y), \quad 1 \leq i \leq m.$$

Below we will abuse notations by writing $a(u) = a(\tau(u))$ and $Y_i(u) = Y_i(\tau(u))$. Notice that the subsystem (3.6b) determines Y as a function of u , and (3.6a) allows us to recover a as a function of u . More precisely, we have

$$(3.7) \quad a(u) = a(0) \exp \left(\int_0^u E(Y(\zeta)) d\zeta \right).$$

Finally, τ can be recovered from integrating $d\tau = a(u)du$, and $b_i(\tau)$ from $a(u)/Y_i(u)$ and the relation between u and τ . Hence given a solution $Y(u)$ of (3.6b) it gives rise to a solution $g_{a,b}(\tau)$ of the backwards Ricci flow. Therefore for circle bundles it suffices for us to focus on the system (3.6b).

Remark 3.1. There is only one point $\xi = (\xi_1, \dots, \xi_m)$ which satisfies $F_i(\xi) = 0$ and $\xi_i > 0$ for all $i = 1, \dots, m$. This actually corresponds to the Einstein metric of connection type and of positive scalar curvature found in [WZ90], which is unique up to homothety among metrics of connection type. To see this, note that $F_i(Y) = 0$ for all i means that $Y_i(p_i - \frac{1}{2}q_i^2 Y_i) = \frac{1}{2}E(Y)$. We may set the right hand side to be Λa where Λ and a are positive constants. With $b_i \doteq a/Y_i$, upon comparison with (1.5) and (1.6) in [WZ90], we obtain the desired conclusion.

We define the nonempty compact convex subset

$$(3.8) \quad \Omega_+ \doteq \{Y \in \mathbb{R}_{\geq 0}^m : F_i(Y) \geq 0 \text{ for each } i = 1, \dots, m\},$$

where the boundary $\partial\Omega_+$ is the union of portions of the level hypersurfaces $F_i(Y) = 0$. We define also the subset

$$\Omega_- \doteq \{Y \in \mathbb{R}_{\geq 0}^m : F_i(Y) \leq 0 \text{ for each } i = 1, \dots, m\}.$$

More generally, let Θ be the collection of nonempty proper subsets of $\{1, 2, \dots, m\}$. For each $\theta \in \Theta$ we define the subsets

$$\Omega_\theta \doteq \{Y \in \mathbb{R}_{\geq 0}^m : F_i(Y) \geq 0 \text{ for each } i \in \theta \text{ and } F_j(Y) \leq 0 \text{ for each } j \notin \theta\}.$$

In short we denote $\Omega_{\{i\}}$ by Ω_i .

The next lemma describes the constant solutions of (3.6b). Let $G(Y)$ denote the vector field $(Y_1 F_1(Y), \dots, Y_m F_m(Y))$, the negative of the vector field in (3.6b).

Lemma 3.2. *The zeros of the vector field $G(Y)$ are*

- (i) *the origin,*
- (ii) *$\{v_\theta, \theta \in \Theta\}$, where v_θ is the only solution which satisfies the equations $F_i(Y) = 0$ and $Y_i > 0$ for $i \in \theta$, and $Y_j = 0$ for $j \notin \theta$. Note that $v_\theta \in \Omega_\theta \setminus \Omega_+$ and $v_\theta \in \Omega_-$.*
- (iii) *the Einstein point ξ described in Remark 3.1. Note that ξ belongs to Ω_+ , Ω_- , and each Ω_θ .*

Proof. The zeros are given by $Y_1 F_1(Y) = \dots = Y_m F_m(Y) = 0$. After a permutation of the indices, we may assume that $Y_1 \neq 0, \dots, Y_k \neq 0, Y_{k+1} = \dots = Y_m = 0$, and $F_1(Y) = \dots = F_k(Y) = 0$, for some $k \in \{0, 1, \dots, m\}$.

If $k = 0$, we are in Case (i), if $k = m$ we are in Case (iii) and $\xi \in \partial\Omega_+$, otherwise we are in one of the situations in Case (ii) where $\theta = \{1, \dots, k\}$. Note that $F_i(Y) \geq 0$ implies that $Y_i \geq 0$.

Now suppose $k \in \{1, \dots, m-1\}$ and $\theta = \{1, \dots, k\}$ in Case (ii). Then $F_j(Y) = -\sum_{l=1}^k n_l q_l^2 Y_l^2 < 0$ for any $j \geq k+1$. So the solution Y lies outside Ω_+ and inside $\Omega_\theta \cap \Omega_-$. To see the existence and the uniqueness of the solution, we notice that the issue is to solve

$$2p_i Y_i - q_i^2 Y_i^2 = E((Y_1, \dots, Y_k)), \quad i = 1, \dots, k$$

with all $Y_i > 0$. This follows from the existence and the uniqueness mentioned in Remark 3.1. \square

To facilitate the study of the global dynamical behavior of the nonlinear system (3.6b), we need to consider the linearization of vector field $G(Y)$ at each of the fixed points $0, v_\theta, \xi$ and a monotonicity formula for solution $Y(u)$.

Lemma 3.3. *Let matrix \mathcal{L}_Y denote the linearization of $G(Y)$ at Y . Then*

- (i) \mathcal{L}_ξ is diagonalizable with one negative and $m - 1$ positive eigenvalues. An eigenvector of the negative eigenvalue can be chosen to have all positive entries.
- (ii) For each nontrivial subset $\theta \subset \{1, \dots, m\}$, \mathcal{L}_{v_θ} has $m - |\theta| + 1$ negative eigenvalues and $|\theta| - 1$ positive eigenvalues.
- (iii) \mathcal{L}_0 is zero matrix $0_{m \times m}$.

Proof. (i) Note that \mathcal{L}_ξ is the matrix

$$\begin{bmatrix} 2\xi_1(p_1 - (n_1 + 1)q_1^2\xi_1) & -2n_2q_2^2\xi_1\xi_2 & \cdots & -2n_mq_m^2\xi_1\xi_m \\ -2n_1q_1^2\xi_2\xi_1 & 2\xi_2(p_2 - (n_2 + 1)q_2^2\xi_2) & \cdots & -2n_mq_m^2\xi_2\xi_m \\ \vdots & \vdots & \ddots & \vdots \\ -2n_1q_1^2\xi_m\xi_1 & -2n_2q_2^2\xi_m\xi_2 & \cdots & 2\xi_m(p_m - (n_m + 1)q_m^2\xi_m) \end{bmatrix}.$$

Define the matrix

$$\beta \doteq \begin{bmatrix} (2n_1 + 1)q_1^2\xi_1^2 & 2n_2q_2^2\xi_1\xi_2 & \cdots & 2n_mq_m^2\xi_1\xi_m \\ 2n_1q_1^2\xi_2\xi_1 & (2n_2 + 1)q_2^2\xi_2^2 & \cdots & 2n_mq_m^2\xi_2\xi_m \\ \vdots & \vdots & \ddots & \vdots \\ 2n_1q_1^2\xi_m\xi_1 & 2n_2q_2^2\xi_m\xi_2 & \cdots & (2n_m + 1)q_m^2\xi_m^2 \end{bmatrix}.$$

Then we have $\mathcal{L}_\xi = E(\xi)I_{m \times m} - \beta$. Define the diagonal matrix $D_\xi \doteq [\xi_1, \dots, \xi_m]$ and the matrix of positive entries $\alpha \doteq D_\xi^{-1}\beta D_\xi$. We have

$$\alpha \doteq \begin{bmatrix} (2n_1 + 1)q_1^2\xi_1^2 & 2n_2q_2^2\xi_1^2\xi_2 & \cdots & 2n_mq_m^2\xi_1^2\xi_m \\ 2n_1q_1^2\xi_1^2\xi_2 & (2n_2 + 1)q_2^2\xi_2^2 & \cdots & 2n_mq_m^2\xi_2^2\xi_m \\ \vdots & \vdots & \ddots & \vdots \\ 2n_1q_1^2\xi_1^2\xi_m & 2n_2q_2^2\xi_2^2\xi_m & \cdots & (2n_m + 1)q_m^2\xi_m^2 \end{bmatrix}.$$

Note that the matrix α and β have the same eigenvalues. We denote by $\lambda_1(\alpha)$ the largest eigenvalue of α . This is known in the literature as the **Perron-Frobenius eigenvalue**.

Let $a_i = 2n_iq_i^2\xi_i^2$ and $\epsilon_i = \frac{1}{2n_i}$. By Lemma 5.1(ii) in the appendix, the matrix α is diagonalizable with eigenvalues $\lambda_i(\alpha)$, $i = 1, \dots, m$. Since the eigenvalues of a matrix depend continuously on the matrix, we may still apply Lemma 5.1(i) to estimate $\lambda_i(\alpha)$ even when not all $\epsilon_i a_i$ are distinct, except that we need to change $<$'s to \leq 's in (5.1).

Note that each of the row sums in α is greater than $2E(\xi)$. Hence the smallest eigenvalue of \mathcal{L}_ξ satisfies $E(\xi) - \lambda_1(\beta) < -E(\xi) < 0$. The corresponding eigenvector of \mathcal{L}_ξ is the eigenvector of β corresponding to $\lambda_1(\beta) = \lambda_1(\alpha)$, which has positive entries by Lemma 5.1(i) and the fact that the entries of D_ξ are positive. The other eigenvalues of α satisfy

$$\min_{i=1, \dots, m} \{q_i^2\xi_i^2\} = \min_{i=1, \dots, m} \{\epsilon_i a_i\} \leq \lambda_j(\alpha) \leq \max_{i=1, \dots, m} \{\epsilon_i a_i\} = \max_{i=1, \dots, m} \{q_i^2\xi_i^2\}$$

where $j = 2, \dots, m$. Hence the corresponding eigenvalues of \mathcal{L}_ξ satisfy

$$\lambda_j(\mathcal{L}_\xi) = E(\xi) - \lambda_j(\beta) \geq E(\xi) - \max_{i=1, \dots, m} \{q_i^2 \xi_i^2\} > 0.$$

(ii) Without loss of generality we may assume $\theta = \{1, \dots, k\}$ with $1 \leq k \leq m-1$. Then \mathcal{L}_{v_θ} is given by

$$\begin{bmatrix} 2Y_1(p_1 - (n_1 + 1)q_1^2 Y_1) & \cdots & -2n_k q_k^2 Y_1 Y_k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -2n_1 q_1^2 Y_k Y_1 & \cdots & 2Y_k(p_k - (n_k + 1)q_k^2 Y_k) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & F_{k+1}(Y) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & F_m(Y) \end{bmatrix}$$

where $Y = v_\theta$. Note that this is a block diagonalized matrix since the submatrix in the upper-left corner is the linearization of the truncated system $(Y_1 F_1(Y), \dots, Y_k F_k(Y))$ at the corresponding Einstein point. By part (i) the submatrix has one negative eigenvalue and $k-1$ positive eigenvalues. Since $F_j(v_\theta) < 0$ for $j = k+1, \dots, m$, (ii) is now proved.

(iii) This is obvious. \square

Now we discuss a monotonicity formula for solution $Y(u)$ of (3.6b). Let $\mathcal{W}(g, f, \tau)$ be Perelman's entropy functional. Let $\mu(g, \tau) \doteq \inf_f \mathcal{W}(g, f, \tau)$ and $\nu(g) \doteq \inf_{\tau > 0} \mu(g, \tau)$. It is well-known that $\nu(g)$ is a monotone quantity under the Ricci flow. For the metrics $g_{a, \vec{b}}$ in (3.1) we define

$$(3.9) \quad \bar{\lambda}(g_{a, \vec{b}}) \doteq \bar{\lambda}(Y) \doteq \left(\prod_{i=1}^m Y_i^{-\frac{2n_i}{n}} \right) \cdot \sum_{i=1}^m (2n_i p_i Y_i - \frac{1}{2} n_i q_i^2 Y_i^2)$$

where $n = 1 + \sum_{i=1}^m 2n_i$ and once again $Y_i = \frac{a}{b_i} > 0$. Note that in our setting $\bar{\lambda}(Y)$ is a smooth function of $Y \in \mathbb{R}_{>0}^m$. If we assume that the minimizing function f of $\inf_f \mathcal{W}(g_{a, \vec{b}}, f, \tau)$ is a constant, then a simple calculation shows that $\nu(g_{a, \vec{b}})$ is a linear function of $\ln(\bar{\lambda}(g_{a, \vec{b}}))$ which would imply the monotonicity of $\bar{\lambda}(g_{a, \vec{b}})$ under the Ricci flow. Actually we can show this monotonicity directly.

Lemma 3.4. *Under ODE (3.6b) we have*

$$(3.10) \quad \begin{aligned} & \left(\prod_{i=1}^m Y_i^{-\frac{2n_i}{n}} \right)^{-1} \frac{d\bar{\lambda}(Y(u))}{du} \\ & \leq - \left(\frac{1}{\sqrt{n}} \left(\sum_{i=1}^m n_i Y_i^2 (2p_i - q_i^2 Y_i)^2 \right)^{1/2} - \sqrt{\frac{n-1}{2n}} E(Y) \right)^2 \leq 0. \end{aligned}$$

Furthermore $\frac{d\bar{\lambda}(Y(u))}{du} = 0$ for some $u = u_0$ if and only if $F_i(Y(u_0)) = 0$ for each i , i.e., $Y(u) = \xi$ for all u .

Proof. We compute

$$\begin{aligned}
 & \left(\prod_{i=1}^m Y_i^{-\frac{2n_i}{n}} \right)^{-1} \frac{d\bar{\lambda}(Y(u))}{du} \\
 &= \left(\frac{4}{n} \sum_{i=1}^m n_i p_i Y_i - \frac{n+1}{n} E(Y) \right) \sum_{i=1}^m \left(2n_i p_i Y_i - \frac{1}{2} n_i q_i^2 Y_i^2 \right) \\
 & \quad + \sum_{i=1}^m \left(-2n_i p_i Y_i F_i(Y) + n_i q_i^2 Y_i^2 F_i(Y) \right) \\
 &= \frac{8}{n} \left(\sum_{i=1}^m n_i p_i Y_i \right)^2 - \frac{2}{n} E(Y) \sum_{i=1}^m n_i p_i Y_i - \frac{n+1}{n} E(Y) \sum_{i=1}^m \left(2n_i p_i Y_i - \frac{1}{2} n_i q_i^2 Y_i^2 \right) \\
 & \quad - \sum_{i=1}^m \left(2n_i p_i Y_i - n_i q_i^2 Y_i^2 \right) (2p_i Y_i - q_i^2 Y_i^2 - E(Y)) \\
 &= \frac{8}{n} \left(\sum_{i=1}^m n_i p_i Y_i \right)^2 - \frac{4}{n} E(Y) \sum_{i=1}^m n_i p_i Y_i - \frac{n-1}{2n} E(Y)^2 - \sum_{i=1}^m n_i (2p_i Y_i - q_i^2 Y_i^2)^2.
 \end{aligned}$$

We have proved

$$\begin{aligned}
 & \left(\prod_{i=1}^m Y_i^{-\frac{2n_i}{n}} \right)^{-1} \frac{d\bar{\lambda}(Y(u))}{du} \\
 (3.11) \quad &= \frac{2}{n} \left(2 \sum_{i=1}^m n_i p_i Y_i - \frac{1}{2} E(Y) \right)^2 - \frac{1}{2} E(Y)^2 - \sum_{i=1}^m n_i Y_i^2 (2p_i - q_i^2 Y_i)^2.
 \end{aligned}$$

Note that by the Cauchy-Schwartz inequality we have

$$\begin{aligned}
 & \left| 2 \sum_{i=1}^m n_i p_i Y_i - \frac{1}{2} E(Y) \right| \leq \left| \sum_{i=1}^m n_i Y_i (2p_i - q_i^2 Y_i) \right| + \frac{1}{2} E(Y) \\
 & \leq \left(\sum_{i=1}^m n_i \right)^{1/2} \left(\sum_{i=1}^m n_i Y_i^2 (2p_i - q_i^2 Y_i)^2 \right)^{1/2} + \frac{1}{2} E(Y).
 \end{aligned}$$

Hence

$$\begin{aligned}
 \left(2 \sum_{i=1}^m n_i p_i Y_i - \frac{1}{2} E(Y) \right)^2 & \leq \left(\frac{n-1}{2} \right) \sum_{i=1}^m n_i Y_i^2 (2p_i - q_i^2 Y_i)^2 + \frac{1}{4} E(Y)^2 \\
 & \quad + \sqrt{\frac{n-1}{2}} E(Y) \left(\sum_{i=1}^m n_i Y_i^2 (2p_i - q_i^2 Y_i)^2 \right)^{1/2}.
 \end{aligned}$$

Combining this inequality with (3.11) we get (3.10).

To have $\frac{d\bar{\lambda}(Y(u))}{du} = 0$ at some u_0 , we need

$$(3.12) \quad \sum_{i=1}^m n_i Y_i^2(u_0) (2p_i - q_i^2 Y_i(u_0))^2 = \frac{n-1}{2} E(Y(u_0))^2$$

and for some constant $c > 0$

$$(3.13) \quad (\sqrt{n_1} Y_1(u_0)(2p_1 - q_1^2 Y_1(u_0)), \dots, \sqrt{n_m} Y_m(u_0)(2p_m - q_m^2 Y_m(u_0))) \\ = c(\sqrt{n_1}, \dots, \sqrt{n_m}),$$

the latter condition owing to the use of the Cauchy-Schwartz inequality. Equation (3.13) gives $2p_i Y_i(u_0) - q_i^2 Y_i^2(u_0) = c$. Substituting this into (3.12) we get $c^2 = E(Y(u_0))^2$. Hence we have proved $F_i(Y(u_0)) = 2p_i Y_i(u_0) - q_i^2 Y_i(u_0)^2 - E(Y(u_0)) = c - c = 0$. \square

3.2. Existence of ancient solutions on circle bundles. In this subsection we show the existence of the ancient solutions of the Ricci flow on circle bundles and derive the asymptotic properties of the corresponding functions a and b_i . We begin with the global dynamical behavior of the system (3.6b).

Lemma 3.5. (i) *If $Y(0) \in \mathbb{R}_{\geq 0}^m$, then the solution $Y(u) \in \mathbb{R}_{\geq 0}^m$ for all $u \in [0, u_*)$ where u_* is the maximal time of existence, and if $Y(u) \in \mathbb{R}_{> 0}^m$ holds for some $u = u_0$ then it holds for all u .*

(ii) *If for some $Y(0) \in \mathbb{R}_{> 0}^m$ there are i and $u_1 \in [0, u_*)$ such that $Y_i(u_1) > \frac{2p_i}{(n_i+1)q_i^2}$, then $u_* < \infty$ and $Y(u)$ approaches infinity as $u \rightarrow u_*^-$. The corresponding solution of the Ricci flow is not ancient.*

(iii) *If $Y(u) \in \mathbb{R}_{\geq 0}^m$ is a solution satisfying $Y_i(u) \leq \frac{2p_i}{(n_i+1)q_i^2}$ for all $u \geq 0$, then $Y(u)$ is defined on $[0, \infty)$. Furthermore either $\lim_{u \rightarrow \infty} Y(u)$ equals to ξ and the corresponding solution of the Ricci flow is ancient, or $Y(u)$ approaches some coordinate hyperplane $Y_i = 0$ as $u \rightarrow \infty$.*

Proof. (i) Note that if $Y_i(u_1) = 0$ for some i and u_1 , then $Y_i(u) = 0$ for all u . Now (i) follows from the uniqueness of solutions of the ODE (3.6b).

(ii) By (3.6b) we have

$$(3.14) \quad \frac{dY_i}{du} \geq Y_i^2((n_i + 1)q_i^2 Y_i - 2p_i).$$

Let a and b be two positive constants. The ODE

$$\frac{dz}{du} = z^2(az - b), \quad \text{with initial condition } z(u_1) > \frac{b}{a}$$

has solution

$$u = \frac{1}{bz} + \frac{a}{b^2} \ln \left(1 - \frac{b}{az} \right) + c < \frac{1}{bz} + \frac{a}{b^2} \left(-\frac{b}{az} - \frac{b^2}{2a^2 z^2} \right) + c = -\frac{1}{2az^2} + c$$

where the constant $c \in (u_1, \infty)$ is determined by the initial condition. With $a = (n_i + 1)q_i^2$, $b = 2p_i$, and $z(u_1) = Y_i(u_1)$, upon comparing (3.14) with the equation for z , we obtain

$$(3.15) \quad Y_i(u)^2 \geq z(u)^2 > \frac{1}{2a(c - u)}, \quad u \in [u_1, u_*).$$

This estimate implies $u_* < \infty$. By the standard extendibility theory we conclude that $Y(u)$ approaches ∞ when $u \rightarrow u_*^-$.

From (3.2b) we conclude that for some i the positive function $b_i(\tau)$ approaches zero in finite τ -time. Hence the corresponding solution $g_{a,\vec{b}}(\tau)$ is not ancient.

(iii) Since $Y(u)$ stays in the bounded set $\{Y, 0 \leq Y_i \leq \frac{2p_i}{(n_i+1)q_i^2}, i = 1, \dots, m\}$ for all $u \in [0, u_*)$, by the extendibility theory of ODEs we conclude that $u_* = \infty$.

We regard $Y(u)$ as a flow line of the vector field $-G(Y)$ from (3.6b) and consider ω_Y , its ω -limit set. Since $Y(u)$ stays in a compact set, ω_Y must be non-empty, compact, connected, and flow-invariant (see [PaM82, Proposition 1.4] or Theorems VII.1.1 and VII.1.2 in [Hr64]). To prove the second part of (iii) it suffices to show that if ω_Y contains a point in $\mathbb{R}_{>0}^m$, say ζ , then $\omega_Y = \{\zeta\} = \{\xi\}$. To see this, suppose $Y(u_i) \rightarrow \zeta$ for some sequence $u_i \rightarrow \infty$, then the monotone quantity $\bar{\lambda}(Y(u_i)) \rightarrow \bar{\lambda}(\zeta)$. Let $Y_*(u)$ be the solution of (3.2b) with initial condition $Y_*(0) = \zeta$. Then $Y_*(u)$ is contained in ω_Y for each u and $Y_*(u)$ is the limit of $Y(u_i + u)$. Since by Lemma 3.4 function $\bar{\lambda}(Y(u))$ is monotone non-increasing, we conclude that $\bar{\lambda}(Y_*(u))$ is a constant function. By the equality statement in Lemma 3.4 we get $Y_*(u) = \xi$. Since ω_Y is connected, we get $\omega_Y = \{\xi\}$.

When $\omega_Y = \{\xi\}$, to see the corresponding solution of the Ricci flow is ancient, we need to prove that $\tau(u) \rightarrow \infty$ as $u \rightarrow \infty$. By (3.2a) $a(\tau)$ is an increasing function of τ , and it follows from the definition of $u(\tau)$ in (3.4) that the inverse function $\tau(u)$ is also increasing. Hence fixing some $u_0 > 0$ and letting $\tau_0 \doteq \tau(u_0)$, we have $a(\tau(u)) \geq a(\tau_0)$ for $u \in [u_0, \infty)$. It follows from $d\tau = a(\tau(u)) du$ that

$$(3.16) \quad \tau(u) - \tau_0 \geq a(\tau_0)(u - u_0), \quad u \in [u_0, \infty).$$

Hence $\lim_{u \rightarrow \infty} \tau(u) = \infty$ and the functions $a(\tau)$ and $b_i(\tau)$ exist for all $\tau \in [0, \infty)$. \square

Now we give some sufficient conditions for the existence of ancient solutions which converge to 0 or to ξ .

Theorem 3.6. *Let $Y(u) \in \mathbb{R}_{\geq 0}^m$ be a solution of (3.6b).*

- (i) *There is a constant $c_0 > 0$ such that if $\sum_{i=1}^m Y_i(0) \leq c_0$, then the solution $Y(u)$ satisfies $\lim_{u \rightarrow \infty} Y(u) = 0$. If $Y(u)$ further satisfies $Y(u) \in \mathbb{R}_{>0}^m$, then the corresponding $g_{a,b}(\tau)$ is an ancient solution of the Ricci flow on circle bundle P_Q .*
- (ii) *There is a C^0 -family of solutions $Y(u)$ with $(m-2)$ -parameters which satisfy $\lim_{u \rightarrow \infty} Y(u) = \xi$. Each corresponding $g_{a,b}(\tau)$ is an ancient solution of the Ricci flow.*
- (iii) *For each nontrivial subset $\theta \subset \{1, \dots, m\}$ there is a C^0 -family of ancient solutions $Y(u)$ with $(|\theta| - 2)$ -parameters which satisfy $\lim_{u \rightarrow \infty} Y(u) = v_\theta$ and $Y_k(u) = 0$ for $k \notin \theta$. But they do not produce any ancient solution of the Ricci flow on P_Q .*

Proof. (i) We assume $Y(0) \neq 0$ and compute that

$$\frac{d}{du} \sum_{i=1}^m Y_i = -2 \sum_{i=1}^m p_i Y_i^2 + \sum_{i=1}^m q_i^2 Y_i^3 + E(Y) \sum_{i=1}^m Y_i < - \sum_{i=1}^m p_i Y_i^2 \leq -\frac{1}{m} \left(\sum_{i=1}^m Y_i \right)^2.$$

The strict inequality above holds when $|Y|$ is sufficiently small since the positive terms in the expression on the left are of third order in Y . The resulting differential inequality then implies that $\lim_{u \rightarrow \infty} \sum_{j=1}^m Y_j = 0$ and (i) follows.

(ii) The existence of the family follows from Hartman-Grobman theorem (see [PaM82, p.59]) and Lemma 3.3(i). The second part of (ii) follows from Lemma 3.5(iii).

(iii) The existence of the family follows from Hartman-Grobman theorem and Lemma 3.3(ii). Since $Y_k(u) = 0$ for $k \notin \theta$, $Y(u)$ does not give rise to any metric $g_{a,b}(\tau)$ on P_Q . Note however that it does give rise to an ancient solution of the Ricci

flow on a corresponding circle bundle over the product of those KE factors whose indices lie in θ . \square

If we pass to the metric tensors of the ancient solutions in Theorem 3.6(i) and (ii), then we gain an extra parameter coming from the initial value of $a(\tau)$ (cf the statement of the Main Theorem in the Introduction). The asymptotic behavior of these ancient solutions as $\tau \rightarrow \infty$ is described by

Theorem 3.7. *Let $g_{a,\bar{b}}(\tau)$ be the ancient solution of the Ricci flow on the circle bundle P_Q given by Theorem 3.6(i). Then*

(i) *for each i and $\tau \in [0, \infty)$, we have*

$$\frac{2n_i p_i}{n_i + 1} \tau + b_i(0) \leq b_i(\tau) \leq 2p_i \tau + b_i(0).$$

(ii) *$\lim_{\tau \rightarrow \infty} a(\tau)$ exists and is positive. Geometrically, as $\tau \rightarrow \infty$, the radii of the circle fibres of the circle bundle P_Q , equipped with the metric $g_{a,\bar{b}}(\tau)$, increase monotonically to a finite value. The length-scale on the KE base factors grows like $\sqrt{\tau}$;*

(iii) *As $\tau \rightarrow \infty$, the rescaled metrics $\tau^{-1} g_{a,\bar{b}}(\tau)$ on P_Q collapse (in the Gromov-Hausdorff distance sense) to the Einstein product metric $2 \sum_i p_i g_i$ on the base $M_1 \times \cdots \times M_m$.*

Proof. (i) For solution $Y(u)$ in Theorem 3.6(i) with $Y_i(u) > 0$, by Lemma 3.5 (ii) we have that for all u

$$q_i^2 Y_i(u) \leq \frac{2p_i}{n_i + 1},$$

and hence by (3.2b) we get

$$\frac{2n_i p_i}{n_i + 1} \leq \frac{db_i}{d\tau} \leq 2p_i.$$

(i) now follows.

(ii) By Theorem 3.6(i) we have $\lim_{\tau \rightarrow \infty} \frac{a(\tau)}{b_i(\tau)} = \lim_{u \rightarrow \infty} Y(u) = 0$ for each i . Fixing an $\epsilon > 0$ to be chosen later, there is a $\tau_0 \geq 0$ such that $\frac{a(\tau)}{b_i(\tau)} < \epsilon$ for each i and $\tau > \tau_0$. By (i) we have

$$(3.17) \quad \frac{1}{a(\tau)} > \frac{((2p_i + 1)\epsilon)^{-1}}{\tau}, \quad \text{for } \tau \geq \max\{\tau_0, b_1(0), \dots, b_m(0)\} \doteq \tau_1.$$

By (3.2a) and (i) we get

$$\frac{1}{a} \frac{da}{d\tau} \leq \sum_{i=1}^m \frac{(n_i + 1)^2 q_i^2}{4n_i p_i^2} \cdot \frac{1}{(\tau + c_i)^2} \quad \text{for } \tau \geq \tau_1,$$

where $c_i = \frac{b_i(0)(n_i + 1)}{2n_i p_i}$. Integrating this inequality over $[\tau_1, \tau]$ we get

$$\begin{aligned} -\frac{1}{a(\tau)} + \frac{1}{a(\tau_1)} &\leq \sum_{i=1}^m \frac{(n_i + 1)^2 q_i^2}{4n_i p_i^2} \left(-\frac{1}{\tau + c_i} + \frac{1}{\tau_1 + c_i} \right) \\ &\leq \left(\sum_{i=1}^m \frac{(n_i + 1)^2 q_i^2}{4n_i p_i^2} \right) \left(\frac{1}{\tau_1} \right). \end{aligned}$$

Hence by (3.17) we have that for $\tau > \tau_1$

$$\frac{1}{a(\tau)} \geq \left(\frac{1}{(2p_i + 1)\epsilon} - \sum_{i=1}^m \frac{(n_i + 1)^2 q_i^2}{4n_i p_i^2} \right) \left(\frac{1}{\tau_1} \right).$$

If we choose ϵ small enough, we have proved that $a(\tau)$ is a bounded increasing function and hence $\lim_{\tau \rightarrow \infty} a(\tau)$ is finite and positive.

(iii) By (ii) we have $\lim_{\tau \rightarrow \infty} \frac{a(\tau)}{\tau} = 0$. Since $\lim_{u \rightarrow \infty} Y(u) = 0$, by (3.2b) we have $\lim_{\tau \rightarrow \infty} \frac{db_i(\tau)}{d\tau} = 2p_i$. Given any $\varepsilon > 0$, there is a τ_0 such that $|\frac{db_i(\tau)}{d\tau} - 2p_i| \leq \varepsilon$ for $\tau > \tau_0$. Hence for τ large enough we have

$$\begin{aligned} \left| \frac{b_i(\tau)}{\tau} - 2p_i \right| &= \left| \frac{b_i(\tau_0) - 2p_i\tau_0 + \int_{\tau_0}^{\tau} \left(\frac{db_i(s)}{ds} - 2p_i \right) ds}{\tau} \right| \\ &\leq \frac{|b_i(\tau_0) - 2p_i\tau_0|}{\tau} + \frac{\int_{\tau_0}^{\tau} \left| \frac{db_i(s)}{ds} - 2p_i \right| ds}{\tau} \\ &\leq \varepsilon + \varepsilon. \end{aligned}$$

Hence $\lim_{\tau \rightarrow \infty} \frac{b_i(\tau)}{\tau} = 2p_i$. (iii) follows and the theorem is proved. \square

Theorem 3.8. *Let $g_{a,\vec{b}}(\tau)$ be the ancient solution of the Ricci flow on the circle bundle P_Q given by Theorem 3.6(ii). Then for any $\epsilon > 0$ small enough there is a $\tau_0 > 0$ such that*

$$(3.18a) \quad (2p_i - q_i^2(\xi_i + \epsilon))(\tau - \tau_0) \leq b_i(\tau) - b_i(\tau_0) \leq (2p_i - q_i^2(\xi_i - \epsilon))(\tau - \tau_0),$$

$$(3.18b) \quad (E(\xi) - \epsilon)(\tau - \tau_0) \leq a(\tau) - a(\tau_0) \leq (E(\xi) + \epsilon)(\tau - \tau_0),$$

for $\tau \geq \tau_0$. Hence

$$(3.19) \quad \lim_{\tau \rightarrow \infty} \frac{a(\tau)}{\tau} = E(\xi), \quad \lim_{\tau \rightarrow \infty} \frac{b_i(\tau)}{\tau} = E(\xi)\xi_i^{-1}.$$

Geometrically, as $\tau \rightarrow \infty$, the circle bundle P_Q , equipped with the metric $\tau^{-1}g_{a,\vec{b}}(\tau)$, converges to a multiple of the Einstein metric corresponding to ξ .

Proof. Since $\lim_{u \rightarrow \infty} Y(u) = \xi$, given any $\epsilon > 0$ small enough we may choose u_0 such that $0 < \xi_i - \epsilon \leq Y_i(u) \leq \xi_i + \epsilon$ for each i and $u \geq u_0$. Let $\tau_0 = \tau(u_0)$, then we get by (3.2b) that for $\tau \geq \tau_0$

$$2p_i - q_i^2(\xi_i + \epsilon) \leq \frac{db_i}{d\tau} \leq 2p_i - q_i^2(\xi_i - \epsilon),$$

from which (3.18a) follows.

Since $\lim_{\tau \rightarrow \infty} \frac{da}{d\tau} = E(\xi)$, given $\epsilon > 0$, by choosing u_0 larger if necessary, we may assume further that $E(\xi) - \epsilon \leq \frac{da}{d\tau} \leq E(\xi) + \epsilon$ for $\tau \geq \tau_0$. (3.18b) now follows. \square

3.3. Curvature properties of the ancient solutions on circle bundles. In this subsection we consider the curvature and some other geometric properties of the ancient solutions in Theorem 3.6(i) and (ii) near $\tau = \infty$.

Theorem 3.9. *Let $g_{a,\bar{b}}(\tau)$ be the ancient solution of the Ricci flow in Theorem 3.6(i).*

(i) *$g_{a,\bar{b}}(\tau)$ is of type I as $\tau \rightarrow \infty$, i.e., there is a constant $C < \infty$ such that for τ large*

$$\tau \cdot \sup_{x \in P_Q} |\text{Rm}_{g_{a,\bar{b}}(\tau)}(x)|_{g_{a,\bar{b}}(\tau)} \leq C.$$

Note that in Theorem 3.7(iii) we have proved the collapsing of the type I rescaled metric $\tau^{-1}g_{a,\bar{b}}(\tau)$ on P_Q .

(ii) *for any $\kappa > 0$, the solution $g_{a,\bar{b}}(\tau)$ is not κ -noncollapsed at all scales.*

Proof. (i) Recall that the metrics $g_{a,\bar{b}}(\tau) = a(\tau)\sigma(\cdot) \otimes \sigma(\cdot) + \sum_i b_i(\tau) g_i$ from (3.1) are Riemannian submersion type metrics with totally geodesic fibres. In the rest of the proof we will drop the subscripts in $g_{a,\bar{b}}$ in order to make notation less cumbersome. We will also take as background metric $g_0 := \sigma(\cdot) \otimes \sigma(\cdot) + \sum_i g_i$, and choose a g_0 -orthonormal basis $\{e_0, e_1^{(1)}, \dots, e_{2n_1}^{(1)}, e_1^{(2)}, \dots, e_{2n_m}^{(m)}\}$ where e_0 is tangent to the fibres and $e_j^{(i)}$ are basic horizontal lifts of tangent vectors to the i th factor of the base. Then the corresponding $g(\tau)$ -orthonormal basis is

$$(3.20) \quad \left\{ \tilde{e}_0 \doteq \frac{e_0}{\sqrt{a(\tau)}}, \frac{e_1^{(1)}}{\sqrt{b_1(\tau)}}, \dots, \frac{e_{2n_1}^{(1)}}{\sqrt{b_1(\tau)}}, \frac{e_1^{(2)}}{\sqrt{b_2(\tau)}}, \dots, \frac{e_{2n_m}^{(m)}}{\sqrt{b_m(\tau)}} \right\}.$$

In analysing the curvature tensor of $g(\tau)$, we use the formulas given in [Bes87, p.241]. Since the fibres are totally geodesic and 1-dimensional, as in [WZ90], it suffices to consider only the following components of $\text{Rm}(g(\tau))$:

$$(3.21a) \quad g(R_{X,\tilde{e}_0}(Y), \tilde{e}_0) = g((\nabla_{\tilde{e}_0} A)_X Y, \tilde{e}_0) + g(A_X \tilde{e}_0, A_Y \tilde{e}_0),$$

$$(3.21b) \quad g(R_{X,Y}(Z), \tilde{e}_0) = g((\nabla_Z A)_X Y, \tilde{e}_0),$$

$$(3.21c) \quad \begin{aligned} g(R_{X,Y}(Z), W) &= g^*(R^*(X, Y)Z, W) - 2g(A_X Y, A_Z W) \\ &\quad + g(A_Y Z, A_X W) - g(A_X Z, A_Y W), \end{aligned}$$

where X, Y, Z, W are horizontal vectors, ∇ is the Levi-Civita connection of $g(\tau)$, and R^* is the curvature tensor of the base metric $g^*(\tau) = \sum_i b_i(\tau) g_i$. Furthermore, A is the O'Neill $(2, 1)$ -tensor for the Riemannian submersion, which has the properties (a) $A_X \tilde{e}_0$ is the horizontal component of $\nabla_X \tilde{e}_0$, and (b) $2A_X Y$ is the vertical component of Lie bracket $[X, Y]$ (and up to a sign is the connection form σ).

For term $g(A_X Y, A_Z W)$ in (3.21c) we compute

$$\begin{aligned} g(A_X Y, A_Z W) &= g(A_X Y, \frac{e_0}{\sqrt{a}}) + g(\frac{e_0}{\sqrt{a}}, A_Z W) \\ &= \frac{1}{4a} g(F(X, Y), e_0) \cdot g(F(Z, W), e_0) \\ &= \frac{a(\tau)}{4} \left(\sum_i q_i \omega_i(X, Y) \right) \cdot \left(\sum_j q_j \omega_j(Z, W) \right), \end{aligned}$$

where we have used [Bes87, 9.54(c)] to get the second equality and F is the curvature form of our connection σ . The term $\omega_i(X, Y)$ is nonzero only if X and Y are tangent to M_i . Hence if X, Y, Z, W are among the $g(\tau)$ -orthonormal basis vectors in (3.20),

we have

$$g(A_X Y, A_Z W) \sim O\left(\frac{a(\tau)}{b_i(\tau)b_j(\tau)}\right) \sim O(\tau^{-2}) \quad \text{as } \tau \rightarrow \infty.$$

The term $g^*(R^*(X, Y)Z, W)$ in (3.21c) is nonzero only if X, Y, Z, W are tangent to M_k for some k . In this case it is the $(0, 4)$ -curvature tensor of metric $b_k(\tau)g_k$. Hence if X, Y, Z, W occur among the $g(\tau)$ -orthogonal basis in (3.20), we have

$$g^*(R^*(X, Y)Z, W) \sim O\left(\frac{1}{b_i(\tau)}\right) \sim O(\tau^{-1}) \quad \text{as } \tau \rightarrow \infty.$$

Let $\{\tilde{e}_l\}$ be the set of all horizontal vectors from (3.20). The term $g(A_X \tilde{e}_0, A_Y \tilde{e}_0)$ in (3.21a) can be evaluated as follows.

$$\begin{aligned} g(A_X \tilde{e}_0, A_Y \tilde{e}_0) &= \frac{1}{a(\tau)} \sum_l g(A_X e_0, \tilde{e}_l) \cdot g(\tilde{e}_l, A_Y e_0) \\ &= \frac{1}{a(\tau)} \sum_l g(e_0, A_X \tilde{e}_l) \cdot g(e_0, A_Y \tilde{e}_l) \\ &= \frac{1}{4a(\tau)} \sum_l g(e_0, F(X, \tilde{e}_l)) \cdot g(e_0, F(Y, \tilde{e}_l)) \\ &= \frac{a(\tau)}{4} \sum_l \left(\sum_i q_i \omega_i(X, \tilde{e}_l) \right) \cdot \left(\sum_j q_j \omega_j(Y, \tilde{e}_l) \right), \end{aligned}$$

where we have used ([Bes87, 9.21d]) to get the second equality. The product term $\omega_i(X, \tilde{e}_l)\omega_j(Y, \tilde{e}_l)$ from the last line is nonzero if both X and Y are tangent to some M_k . Hence if X, Y occur among the $g(\tau)$ -orthogonal basis in (3.20), then

$$g(A_X \tilde{e}_0, A_Y \tilde{e}_0) \sim O\left(\frac{a(\tau)}{b_k^2(\tau)}\right) \sim O(\tau^{-2}) \quad \text{as } \tau \rightarrow \infty.$$

The term $g((\nabla_Z A)_X Y, \tilde{e}_0)$ in (3.21b) equals to zero ([WZ90, p.243]). This is essentially the covariant derivative of F .

Finally the term $g((\nabla_{\tilde{e}_0} A)_X Y, \tilde{e}_0)$ in (3.21a) can be computed as follows ([WZ90, p.243]):

$$\begin{aligned} g((\nabla_{\tilde{e}_0} A)_X Y, \tilde{e}_0) &= \frac{1}{a(\tau)} g((\nabla_{e_0} A)_X Y, e_0) \\ &= \frac{1}{a(\tau)} (-g(\mathcal{L}_{e_0} X, \mathcal{L}_{e_0} Y) + g(\mathcal{L}_{e_0} Y, \mathcal{L}_{e_0} X)) \\ (3.22) \quad &= 0, \end{aligned}$$

where \mathcal{L}_{e_0} is, up to a factor of $\frac{1}{2}$, the skew symmetric operator corresponding to the curvature 2-form via the metric g .

Hence

$$\sup_{x \in P} \|\text{Rm}_{g(\tau)}(x)\|_{g(\tau)}^2 = \sup_x \sum_{l_1, l_2, l_3, l_4} R(\tilde{e}_{l_1}, \tilde{e}_{l_2}, \tilde{e}_{l_3}, \tilde{e}_{l_4})^2(x) \sim O(\tau^{-2}) \quad \text{as } \tau \rightarrow \infty,$$

where we recall that $\{\tilde{e}_l\}$ consists of all the $g(\tau)$ -orthonormal horizontal vectors from (3.20). This proves our first assertion that the ancient solution $g_{a,\vec{b}}(\tau)$ is of type I as $\tau \rightarrow \infty$.

(ii) By (i) and scaling we know that the metric $\tilde{g}(\tau) \doteq \frac{1}{\tau} g(\tau)$ has bounded curvature for $\tau \geq 1$. On the other hand the volume is given by

$$\begin{aligned} \text{Vol}_{\tilde{g}(\tau)}(P) &= \frac{1}{\tau^{(1+\sum_{i=1}^m 2n_i)/2}} \text{Vol}_{g(\tau)}(P) \\ &\sim \tau^{-(1+\sum_{i=1}^m 2n_i)/2} \cdot \tau^{\sum_{i=1}^m n_i} \text{Vol}_{g_0}(P) \\ &= \tau^{-1/2} \text{Vol}_{g_0}(P) \rightarrow 0 \end{aligned}$$

as $\tau \rightarrow \infty$. Hence $\tilde{g}(\tau)$ *cannot* be κ -noncollapsed at all scales (uniformly in τ).

This proves the theorem. \square

Remark 3.10. (i) Note that from (2.4) the nonzero components of the Ricci tensor of $g_{a,\vec{b}}(\tau)$ are given by

$$\begin{aligned} \text{Rc}_{g_{a,\vec{b}}(\tau)}(\tilde{e}_0, \tilde{e}_0) &= \frac{1}{2} \sum_i n_i q_i^2 \frac{a(\tau)}{b_i(\tau)^2} \sim O(\tau^{-2}), \\ \text{Rc}_{g_{a,\vec{b}}(\tau)}\left(\frac{e_j^{(i)}}{\sqrt{b_i(\tau)}}, \frac{e_j^{(i)}}{\sqrt{b_i(\tau)}}\right) &= \frac{p_i}{b_i(\tau)} - \frac{q_i^2}{2} \frac{a(\tau)}{b_i(\tau)^2} \sim O(\tau^{-1}) \end{aligned}$$

as $\tau \rightarrow \infty$. By Lemma 3.5(ii) we have $Y_i \leq 2p_i/((n_i + 1)q_i^2)$, $i = 1, \dots, m$, for any ancient solutions in Theorem 3.6(i) and (ii). Hence these solutions all have positive Ricci curvature over their interval of existence. For completeness we record the formula for scalar curvature below,

$$R_{g_{a,\vec{b}}(\tau)} = \frac{1}{a(\tau)} \sum_{i=1}^m \left(2n_i p_i \frac{a(\tau)}{b_i(\tau)} - \frac{1}{2} n_i q_i^2 \frac{a(\tau)^2}{b_i(\tau)^2} \right).$$

(ii) About the sectional curvatures of $g_{a,\vec{b}}(\tau)$, note that by (3.21a) and (3.22) we know that $K(e_0 \wedge X)$ is always nonnegative for horizontal vectors X . On the other hand $K(X \wedge Y) = K^*(X \wedge Y) - 3|A_X Y|^2$ where K^* is the sectional curvature of metric $g^* = \sum_i b_i(\tau) g_i$. So in general we can have negative sectional curvatures.

Next we analyse the behavior of the other class of ancient solutions in Theorem 3.6.

Theorem 3.11. *Let $g_{a,\vec{b}}(\tau)$ be the ancient solution of the Ricci flow on circle bundle P_Q given in Theorem 3.6(ii).*

(i) *The solution $g_{a,\vec{b}}(\tau)$ is of type I as $\tau \rightarrow \infty$, i.e., there is a constant $C < \infty$ such that for τ large*

$$\tau \cdot \sup_{x \in P_Q} |\text{Rm}_{g_{a,\vec{b}}(\tau)}(x)|_{g_{a,\vec{b}}(\tau)} \leq C.$$

Note that in Theorem 3.8 we have proved that the type I rescaled metrics $\tau^{-1}g_{a,\vec{b}}(\tau)$ on P_Q converge to a multiple of the Einstein metric corresponding to ξ .

(ii) *There is a $\kappa > 0$ so that the solution $g_{a,\vec{b}}(\tau)$ is κ -noncollapsed at all scales.*

Proof. (i) This follows from an inspection of the proof of Theorem 3.9(i) and the linear growth of $b_i(\tau)$ and $a(\tau)$ given by (3.18a)–(3.18b).

(ii) Since P_Q is compact and $g_{a,\vec{b}}(\tau)$ is non-flat metric, there is a $\kappa_1 > 0$ such that metrics $g_{a,\vec{b}}(\tau)$ is κ_1 -noncollapsed for $\tau \in [0, 1]$. Since $\tau^{-1}g_{a,\vec{b}}(\tau)$ converges to an Einstein metric, there is a $v_0 > 0$ such that $\text{Vol}_{\tau^{-1}g_{a,\vec{b}}(\tau)}(B_{\tau^{-1}g_{a,\vec{b}}(\tau)}(x, 1)) \geq v_0$ for all $x \in P_Q$ and $\tau \geq 1$. Since the Einstein metric is non-flat, there is a $\kappa_2 > 0$ such that $\tau^{-1}g_{a,\vec{b}}(\tau)$ is κ_2 -noncollapsed for $\tau \geq 1$. Since κ -noncollapsing is a property preserved by scaling, hence $g_{a,\vec{b}}(\tau)$ is κ_2 -noncollapsed for $\tau \geq 1$. Now (ii) follows. \square

Remark 3.12. Recall that the cohomogeneity of a (compact) Riemannian manifold is the codimension of a principal orbit of the action of its isometry group. Thus for an n -dimensional Riemannian manifold, cohomogeneity n means that its isometry group is discrete. In section 3 of [WZ90], the cohomogeneity of the Einstein metrics constructed there was studied, and it was shown that, provided that the topology of the torus bundles is sufficiently complicated in a suitable sense, then the cohomogeneity of the Einstein metrics is equal to the sum of the cohomogeneities of the KE factors in the base. The same arguments, especially Lemma 3.6 there, can be used to show that the same equation holds for the cohomogeneities of our ancient solutions, i.e.,

$$\text{coh}(P_Q, g_{h,\vec{b}}(\tau)) = \sum_{i=1}^m \text{coh}(M_i, g_i).$$

(Note that $b_i(\tau)g_i$ has the same isometry group as g_i .)

For the convenience of the reader, we give a short summary of the main ideas involved. The isometries of the base of our torus bundles certainly lift to the total spaces because these are isometric toral quotients of the product of the universal covers of the principal circle bundles associated to the anti-canonical line bundles over the base factors. The equation above would then hold if all isometries of the total spaces map the totally geodesic fibres to each other. Lemma 3.6 in [WZ90] gives a lower bound for the least period of all non-trivial closed curves which are projections of periodic geodesics in the total space. Since an isometry must map closed geodesics to closed geodesics, the assumption on the topology of the bundles ensures that least period of the fibre geodesics is less than this lower bound. In such a situation any isometry must be the lift of an isometry from the base or comes from the isometric torus action.

For our ancient solutions described in Theorem 3.7, the metrics $\tau^{-1}g_{a,\vec{b}}(\tau)$ automatically have very short fibres when τ is large and so the arguments sketched above apply without any assumption on the topology of the bundles. (Note that the bounds needed for applying Lemma 3.6 in [WZ90] follow from our bounds for $q_i Y(u)$.) Since an overall scaling does not affect cohomogeneities, the equality between cohomogeneities applies to the metrics $g_{a,\vec{b}}(\tau)$ also for large enough τ .

For our ancient solutions described in Theorem 3.8, when τ is sufficiently large, the metrics $\tau^{-1}g_{a,\vec{b}}(\tau)$ are close to the Einstein metric, and so with the same assumptions on the topology of the bundle, the desired equation for the cohomogeneities hold.

We are then in a position to apply Theorem 1.2 in [Ko10] on the stability of isometry groups under the Ricci flow to get the above equation for all values of τ . Therefore,

our ancient solutions in general can have arbitrary cohomogeneity by suitable choices of the Fano KE factors and, if necessary, the topology of the bundles.

3.4. The forward limits of some ancient solutions on circle bundles. In this subsection we investigate when ancient solutions from Theorem 3.6(i) have the property that $\lim_{u \rightarrow -\infty} Y(u) = \xi$. Actually such ancient solutions lie in Ω_+ . In §3.5 we will consider some ancient solutions from Theorem 3.6(ii) for which $\lim_{u \rightarrow -\infty} Y(u) = v_\theta$. Note that part (i) of the following theorem has some overlap with Theorem 3.6(i).

Theorem 3.13. *Let $a(\tau)$ and $b_i(\tau)$ be solutions of system (3.2a)–(3.2b). Let Ω_+ be the compact region defined in (3.8). Then*

- (i) *for any initial data $a(0) > 0$ and $(b_1(0), \dots, b_m(0))$ which satisfy $\left(\frac{a(0)}{b_1(0)}, \dots, \frac{a(0)}{b_m(0)}\right) \in \Omega_+ \setminus \{0\}$, $a(\tau)$ and $b_i(\tau)$ exist for all τ -time and satisfy $\left(\frac{a(\tau)}{b_1(\tau)}, \dots, \frac{a(\tau)}{b_m(\tau)}\right) \in \Omega_+ \setminus \{0\}$ and $\lim_{\tau \rightarrow \infty} \left(\frac{a(\tau)}{b_1(\tau)}, \dots, \frac{a(\tau)}{b_m(\tau)}\right) = 0$;*
- (ii) *There is only one solution in (i) whose corresponding $Y(u)$ satisfies $\lim_{u \rightarrow -\infty} Y(u) = \xi$. For this solution $\lim_{u \rightarrow -\infty} \tau(u) \doteq -T_1$ is finite, where $\tau(u)$ is defined by (3.4).*

Proof. By the discussion before Remark 3.1 we may work with the system (3.6b) for $Y(u)$.

(i) By assumption we may assume that the point $Y(0) \in \Omega_+$ is not a stationary point of the vector field of (3.6b). To see that the solution $Y(u)$ stays in Ω_+ , fix an index $1 \leq i_0 \leq m$. The gradient of the level hypersurface $F_{i_0} = 0$ pointing into Ω_+ is given by

$$\begin{aligned} \nabla F_{i_0} = & (-2n_1 q_1^2 Y_1, -2n_2 q_2^2 Y_2, \dots, -2n_{i_0-1} q_{i_0-1}^2 Y_{i_0-1}, \\ & 2p_{i_0} - 2(n_{i_0} + 1) q_{i_0}^2 Y_{i_0}, -2n_{i_0+1} q_{i_0+1}^2 Y_{i_0+1}, \dots, -2n_m q_m^2 Y_m). \end{aligned}$$

Taking the inner product of ∇F_{i_0} with vector field $-G(Y)$ defined by the ODE (3.6b), we get

$$\begin{aligned} & 2n_1 q_1^2 Y_1^2 (2p_1 Y_1 - q_1^2 Y_1^2 - E(Y)) + \dots + 2n_m q_m^2 Y_m^2 (2p_m Y_m - q_m^2 Y_m^2 - E(Y)) \\ (3.23) \quad & + (2p_{i_0} - 2q_{i_0}^2 Y_{i_0}) (E(Y) + q_{i_0}^2 Y_{i_0}^2 - 2p_{i_0} Y_{i_0}) Y_{i_0}. \end{aligned}$$

Because $2p_i Y_i - q_i^2 Y_i^2 - E(Y) \geq 0$ in Ω_+ and $E(Y) + q_{i_0}^2 Y_{i_0}^2 - 2p_{i_0} Y_{i_0} = 0$ on the hypersurface $F_{i_0}(Y) = 0$, the quantity in (3.23) is nonnegative and it equals zero if and only if $Y_1 F_1(Y) = \dots = Y_m F_m(Y) = 0$.

Since $Y(u)$ stays in a compact set, by the extendibility theory of ODEs the solution $Y(u)$ exists for all $u \in [0, \infty)$. By Lemma 3.5(iii) and the fact that Ω_+ intersects any coordinate hyperplane only at the origin, the corresponding solution of the Ricci flow is either ancient with $\lim_{u \rightarrow \infty} Y(u) = 0$ or converges to ξ . Below we rule out the second possibility by considering three cases.

(ia) The first case is when there is an i_0 such that $Y_{i_0}(0) < \xi_{i_0}$. From $\frac{dY_{i_0}}{du}(u) = -Y_{i_0}(u)F_{i_0}(Y(u)) \leq 0$ we get $\lim_{u \rightarrow \infty} Y_{i_0}(u) < \xi_{i_0}$. Hence in this case we have $\lim_{u \rightarrow \infty} Y(u) = 0$.

(ib) Next we rule out the case in which $Y(0) \in \Omega_+$ and $Y_i(0) > \xi_i$ for each i . In this situation, we have

$$(3.24) \quad \xi_i < \frac{p_i}{(n_i + 1)q_i^2}, \quad \text{for all } i.$$

To see this we compute

$$\begin{aligned} 0 &\leq F_i(Y(0)) - F_i(\xi) \\ &= (2p_i - (n_i + 1)q_i^2(Y_i(0) + \xi_i)) \cdot (Y_i(0) - \xi_i) - \sum_{j \neq i} n_j q_j^2(Y_j(0) - \xi_j)(Y_j(0) + \xi_j). \end{aligned}$$

Hence $2p_i - (n_i + 1)q_i^2(Y_i(u) + \xi_i) \geq 0$ and (3.24) follows.

Let

$$(3.25) \quad a_i \doteq 2(p_i - (n_i + 1)q_i^2\xi_i) > 0 \quad \text{and} \quad b_i \doteq 2n_i q_i^2 \xi_i > 0.$$

We compute

$$\begin{aligned} \sum_{i=1}^m \frac{b_i}{a_i + b_i} &= \sum_{i=1}^m \frac{2n_i q_i^2 \xi_i^2}{2(p_i \xi_i - (n_i + 1)q_i^2 \xi_i^2) + 2n_i q_i^2 \xi_i^2} \\ &= \sum_{i=1}^m \frac{2n_i q_i^2 \xi_i^2}{(\sum_{j=1}^m n_j q_j^2 \xi_j^2) - q_i^2 \xi_i^2}, \end{aligned}$$

where we have used $F_i(\xi) = 0$ to get the last equality. Hence we get

$$(3.26) \quad \sum_{i=1}^m \frac{b_i}{a_i + b_i} > 2.$$

We may therefore write $\sum_{i=1}^m \frac{b_i}{a_i + b_i} = \frac{1}{1-\alpha}$ for some $\alpha \in (\frac{1}{2}, 1)$.

We now show that there is a convex combination of the gradients $\nabla F_i(\xi)$ which is of the form $-(\delta_1, \dots, \delta_m)$ with each $\delta_i > 0$, i.e., there is a solution $(\lambda_1, \dots, \lambda_m)$ of

$$(3.27) \quad \sum_{i=1}^m \lambda_i \cdot \nabla F_i(\xi) = -(\delta_1, \dots, \delta_m)$$

which satisfies $\sum_{i=1}^m \lambda_i = 1$ with each $\lambda_i > 0$. Note that the equation (3.27) can be written in components as

$$a_i \lambda_i - b_i \sum_{j \neq i} \lambda_j = -\delta_i,$$

which is equivalent to

$$(a_i + b_i) \lambda_i = b_i - \delta_i.$$

It is now trivial to verify that $\delta_i = \alpha b_i$ and $\lambda_i = (1 - \alpha) \frac{b_i}{a_i + b_i}$ are solutions which satisfy all the conditions. This gives (3.27).

Recall that the tangent cone of the convex set Ω_+ at ξ is given by

$$T_\xi \Omega_+ = \{w \in \mathbb{R}^m : \nabla F_i(\xi) \cdot (w - \xi) \geq 0\}.$$

By (3.27) there is a supporting plane of this cone whose inward-pointing normal takes the form $-(\delta_1, \dots, \delta_m)$. We obtain an immediate contradiction that $Y(0)$ would not lie in the tangent cone, since we have $-(\delta_1, \dots, \delta_m) \cdot (Y(0) - \xi) < 0$.

(ic) Finally we consider the case in which there are i_0 and j_0 such that $Y_{i_0}(0) = \xi_{i_0}$, $Y_{j_0}(0) > \xi_{j_0}$, and $Y_i(0) \geq \xi_i$ for all i . It follows that $F_{i_0}(Y(0)) < F_{i_0}(\xi) = 0$. Then $\frac{dY_{i_0}}{du}(0) = -Y_{i_0}(0)F_{i_0}(Y(0)) > 0$ and hence there is a small $u_1 > 0$ such that $Y_i(u_1) > \xi_i$ for all i . Treating u_1 as initial time, we see that this case is impossible by the conclusion of (ib). This finishes the proof of (i).

(ii) By Lemma 3.3(i) we know that exact one eigenvalue of \mathcal{L}_ξ , denoted say by λ_1 , is negative, and that the corresponding eigenvector z can be assumed to have all negative entries. Hence $(\xi_i \nabla F_i|_\xi) \cdot z = \lambda_1 z_i > 0$ for each i . Thus $\nabla F_i|_\xi \cdot z > 0$ and z points into $\text{int}(\Omega_+)$. Applying the Hartman-Grobman theorem we conclude that one side of the one dimensional unstable manifold of vector field $-G(Y)$ at ξ lies in $\text{int}(\Omega_+)$ due to (i). This proves first part of (ii). Note that the longtime existence of $Y(u)$ with $u \rightarrow -\infty$ follows since the solution lies in a compact set.

To see $\lim_{u \rightarrow -\infty} \tau(u)$ is finite, since $\lim_{u \rightarrow -\infty} Y(u) = \xi$, we can choose $u_0 < 0$ small enough so that when $u < u_0$ we have $|Y_i(u) - \xi_i| < \frac{1}{2}\xi_i$ for each i . From (3.2a) we conclude that $\frac{da}{d\tau} > \sum_{i=1}^m \frac{1}{4}n_i q_i^2 \xi_i^2$ when $\tau < \tau(u_0)$ which, we recall from (3.4), satisfies $u_0 = \int_0^{\tau(u_0)} \frac{1}{a(\zeta)} d\zeta$. This implies that there is a finite $-T_1 < \tau(u_0)$ such that $\lim_{\tau \rightarrow -T_1^+} a(\tau) = 0$ and $a(\tau) \geq (\sum_{i=1}^m \frac{1}{4}n_i q_i^2 \xi_i^2)(\tau + T_1)$ for $\tau \in [-T_1, \tau(u_0)]$. Then it follows from

$$\frac{du}{d\tau} = \frac{1}{a(\tau)} \leq \frac{1}{\sum_{i=1}^m \frac{1}{4}n_i q_i^2 \xi_i^2} \cdot \frac{1}{\tau + T_1} \quad \text{for } \tau \in (-T_1, \tau(u_0)]$$

that $\lim_{u \rightarrow -\infty} \tau(u) = -T_1$. \square

Remark 3.14. By the Hartman-Grobman theorem and Theorem 3.13(ii), the other half of the unstable manifold of the vector field $-G(Y)$ at ξ lies inside $\text{int}(\Omega_-)$. Furthermore, none of the eigenvectors of \mathcal{L}_ξ corresponding to positive eigenvalues lie in the interior of the tangent cones $T_\xi \Omega_+$ or $T_\xi \Omega_-$.

In the next two remarks we digress to discuss the backwards Ricci flow in pseudo Riemannian geometry. Readers interested only in the Riemannian Ricci flow can skip to Lemma 3.17.

Remark 3.15. Note that in the proof of Theorem 3.13 what is essential for all the arguments is that we have the same sign for the products $p_i Y_i$. By the next remark and remarks at the bottom of [ONe66, p.465] it follows that the formulas for backwards Ricci flow (3.2a) and (3.2b) hold in the pseudo Riemannian case, i.e., where some of b_i , $i \in I \subset \{1, \dots, m\}$ and possibly a , are negative.

There are two situations. For convenience let us assume (by a translation in τ) that our Riemannian ancient solution is parametrized by the interval $(0, \infty)$. In the first case, we leave $a(\tau)$ positive, and choose a subset $I \subset \{1, \dots, m\}$ and replace the corresponding b_i by $-b_i$. To ensure that the flow equations (3.2a) and (3.2b) remain unchanged we must make the corresponding p_i , $i \in I$ negative, i.e., replace each of the Fano KE base factors $(M_i^{n_i}, g_i)$, $i \in I$, by a KE manifold with negative scalar curvature of the same dimension. We then obtain ancient flows for pseudo Riemannian metrics of signature type $(1 + 2 \sum_{i \notin I} n_i, 2 \sum_{i \in I} n_i)$.

In the second situation, we replace $a(\tau)$ by $-a(\tau)$. In order to leave (3.2a) unchanged, we need to replace τ by $-\tau$. This means that the corresponding Ricci flow is defined on $(0, \infty)$, i.e., it is an immortal flow. As before choose a subset I of $I \subset \{1, \dots, m\}$. This time, for any $i \in I$ we replace b_i by $-b_i$ and leave the corresponding Fano KE factor in the base alone, while for any $i \notin I$, we leave b_i alone and replace the corresponding Fano KE base factor by a KE manifold with negative scalar curvature of the same dimension. Together with the reflection in τ we see that both flow equations remain unchanged. Therefore we obtain immortal flows for pseudo

Riemannian metrics of signature type $(2 \sum_{i \notin I} n_i, 1 + 2 \sum_{i \in I} n_i)$. In particular, for the choice $I = \{1, \dots, m\}$, we have an immortal flow with Lorentzian signature.

Actually the above correspondence for the flows applies also to the Riemannian Einstein metrics on torus bundles over products of Fano KE manifolds and pseudo Riemannian Einstein metrics on torus bundles over products of KE manifolds of either positive or negative scalar curvature.

Remark 3.16. For any local coordinates $\{x^i\}$ on a pseudo Riemannian manifold (W^n, h) , (using the the convention $R(X, Y)Z = \nabla_{[X, Y]}Z - (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z)$), the curvature tensor has the same expression in terms of $h_{ij} = h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, X^i, Y^j , and Z^k as in the Riemannian case. (In the above $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^j \frac{\partial}{\partial x^j}$, and $Z = Z^k \frac{\partial}{\partial x^k}$.) Hence the formula for the Ricci tensor

$$\text{Rc}(X, Z) = \text{tr}(Y \rightarrow R(X, Y)Z) = h^{ij} h(R(X, \frac{\partial}{\partial x^i})Z, \frac{\partial}{\partial x^j})$$

also has the same expression as in Riemannian case.

Let $\pi : (M, g) \rightarrow (B, \tilde{g})$ be a smooth submersion between two pseudo Riemannian manifolds. We assume that for any $p \in M$ the vertical space $V_p \subset (T_p M, g_p)$ is non-degenerate and that $d\pi : T_p M \rightarrow T_{\pi(p)} B$ is an isometry on the horizontal space $H_p \doteq V_p^\perp$, then the calculations for the Ricci tensor in [Bes87] from Theorem 9.28 to Proposition 9.36 go through literally without any change (as was already implied in [ONe66, p.465]).

However if we want to use orthonormal frames to calculate the Ricci tensor for a pseudo Riemannian submersion, we need to define an **orthonormal** frame to be a frame $\{e_i\}$ such that $h(e_i, e_i) \doteq \epsilon_i = \pm 1$ and $h(e_i, e_j) = 0$ for $i \neq j$. In such a frame the trace of a linear map $A : T_p W \rightarrow T_p W$ is given by

$$\text{tr } A = \sum_i \epsilon_i h(A(e_i), e_i).$$

The formula for the Ricci tensor of a pseudo Riemannian submersion is then given by the same expression as in the Riemannian case in terms of the h_{ij} .

The technique in the proof of Theorem 3.13 can be used to improve Lemma 3.5(ii).

Lemma 3.17. *Let $a(\tau)$ and $b_i(\tau)$ be a solution of the system (3.2a)–(3.2b). Suppose the initial data $a(0)$ and $b_i(0)$ satisfy $Y(0) = (\frac{a(0)}{b_1(0)}, \dots, \frac{a(0)}{b_m(0)}) \in \Omega_- \setminus (\{\xi\} \cup \{v_\theta, \theta \in \Theta\})$. Then the corresponding solution $g_{a, \vec{b}}(\tau)$ terminates in finite τ -time and $(\frac{a(\tau)}{b_1(\tau)}, \dots, \frac{a(\tau)}{b_m(\tau)})$ remains in $\Omega_- \setminus (\{\xi\} \cup \{v_\theta, \theta \in \Theta\})$. Let u_* denote the maximal u -time of existence of the corresponding $Y(u)$. Then u_* is finite and $Y(u)$ approaches ∞ as $u \nearrow u_*^-$.*

Proof. Assuming $Y(0) \in \Omega_- \setminus (\{\xi\} \cup \{v_\theta, \theta \in \Theta\})$, by an argument similar to that in the proof of Theorem 3.13(i) one shows that at a boundary point of $\partial\Omega_-$ where $F_{i_0} = 0$ for some i_0 , the vector field $-G(Y)$ points into Ω_- . Hence the solution $Y(u)$ stays in Ω_- as long as it exists. Note that $Y_i(u) \neq 0$ for all i and $u < u_*$.

Next we will prove that the solution $Y(u)$ approaches ∞ as $u \rightarrow u_*^-$. If this does not happen, then since $\frac{dY_i}{du} \geq 0$ for each i , it follows that $Y(u)$ must converge to a

finite point as $u \rightarrow u_*^-$. Since $Y_i(0) > 0$ and $Y(u)$ satisfies (3.6b), the limit of $Y(u)$ must be ξ . There are now three cases to consider.

(a) There is an i_0 such that $Y_{i_0}(0) > \xi_{i_0}$. Then $Y_{i_0}(u)$ cannot approach ξ_{i_0} as $\frac{dY_{i_0}}{du} \geq 0$. This is a contradiction.

(b) $Y_i(0) < \xi_i$ for all i . In this case, since $0 \geq F_i(Y(u)) - F_i(\xi)$, a computation similar to that in the proof of (3.24) gives $2p_i - (n_i + 1)q_i^2(Y_i(u) + \xi_i) \geq 0$. Letting $u \rightarrow u_*^-$, we get

$$\xi_i \leq \frac{p_i}{(n_i + 1)q_i^2}, \quad \text{for all } i.$$

Let a_i and b_i be defined as in (3.25). Calculating as in (3.26) we have $\sum_{i=1}^m \frac{b_i}{a_i + b_i} > 2$. We may therefore write $\sum_{i=1}^m \frac{b_i}{a_i + b_i} = \frac{1}{1-\alpha}$ for some $\alpha \in (\frac{1}{2}, 1)$. Hence in this case (3.27) holds again.

Let $\delta_{\min} \doteq \min\{\delta_1, \dots, \delta_m\} > 0$. Then for any unit vector (w_1, \dots, w_m) with each $w_i < 0$ we have

$$(w_1, \dots, w_m) \cdot (-(\delta_1, \dots, \delta_m)) \geq \delta_{\min} \sum_{i=1}^m (-w_i) \geq \delta_{\min} \sum_{i=1}^m w_i^2 = \delta_{\min}.$$

In particular taking $(w_1, \dots, w_m) = \frac{Y(u) - \xi}{\|Y(u) - \xi\|}$ we have

$$(3.28) \quad \frac{Y(u) - \xi}{\|Y(u) - \xi\|} \cdot (\delta_1, \dots, \delta_m) \leq -\delta_{\min} \quad \text{for } u \in [0, u_*].$$

By (3.27), for any unit vector (w_1, \dots, w_m) in the tangent cone $T_\xi \Omega_-$, we have

$$(w_1, \dots, w_m) \cdot (-(\delta_1, \dots, \delta_m)) \leq 0.$$

Since $Y(u) \in \Omega_-$ approaches ξ as $u \rightarrow u_*^-$, the distance between the unit vector $\frac{Y(u) - \xi}{\|Y(u) - \xi\|}$ and $T_\xi \Omega_-$ approaches zero. Hence for u sufficiently close to u_*^- we have

$$(3.29) \quad \frac{Y(u) - \xi}{\|Y(u) - \xi\|} \cdot (-(\delta_1, \dots, \delta_m)) \leq \frac{1}{2}\delta_{\min}.$$

This contradicts (3.28), hence case (b) is impossible.

(c) There are i_0 and j_0 such that $Y_{i_0}(0) = \xi_{i_0}$, $Y_{j_0}(0) < \xi_{j_0}$, and $Y_i(0) \leq \xi_i$ for all other i . It follows that $F_{i_0}(Y(0)) > F_{i_0}(\xi) = 0$. Then $\frac{dY_{i_0}}{du}(0) = -Y_{i_0}(0)F_{i_0}(Y(0)) < 0$ and hence there is a small $u_1 > 0$ such that $Y_{i_0}(u_1) < \xi_{i_0}$ for all i . Treating u_1 as the initial time, we see that this case is impossible by the conclusion of (b).

Now we have proved $\lim_{u \rightarrow u_*^-} Y(u) = \infty$. By Lemma 3.5(ii) we conclude that the corresponding Ricci flow solution is not ancient. \square

Next we analyse the geometric behavior as τ tends to $-T_1$ for the special solution given in Theorem 3.13(ii).

Theorem 3.18. *Let $g_{a,b}(\tau), \tau \in (-T_1, \infty)$, be the ancient solution of the Ricci flow in Theorem 3.13(ii).*

(i) *We have*

$$\lim_{\tau \rightarrow -T_1^+} a(\tau) = 0, \quad \lim_{\tau \rightarrow -T_1^+} b_i(\tau) = 0, \quad \lim_{\tau \rightarrow -T_1^+} \frac{a(\tau)}{b_i(\tau)} = \xi_i, \quad \lim_{\tau \rightarrow -T_1^+} \frac{a(\tau)}{T_1 + \tau} = E(\xi).$$

Geometrically, as $\tau \rightarrow -T_1^+$, the circle bundles P_Q , equipped with the metric $g_{a,\vec{b}}(\tau)$, collapses to a point in the Gromov-Hausdorff topology.

(ii) The solution $g_{a,\vec{b}}(\tau)$ is of type I as $\tau \rightarrow -T_1^+$, i.e., there is a positive constant $C < \infty$ such that for $\tau \in (-T_1, 0]$ we have

$$(T_1 + \tau) \cdot \sup_{x \in P_Q} |\text{Rm}_{g_{a,\vec{b}}(\tau)}(x)|_{g_{a,\vec{b}}(\tau)} \leq C.$$

(iii) As $\tau \rightarrow -T_1^+$ the rescaled metric $(T_1 + \tau)^{-1}g_{a,\vec{b}}(\tau)$ on P_Q converges to the Einstein metric $E(\xi)g_{1,(\xi_1^{-1}, \dots, \xi_m^{-1})}$, where $g_{1,(\xi_1^{-1}, \dots, \xi_m^{-1})} = \sigma(\cdot) \otimes \sigma(\cdot) + \sum_i \xi_i^{-1} g_i$. The metric $g_{1,(\xi_1^{-1}, \dots, \xi_m^{-1})}$ has Ricci tensor $\text{Rc} = \frac{E(\xi)}{2} \cdot g_{1,(\xi_1^{-1}, \dots, \xi_m^{-1})}$ and scalar curvature $R = 2 \sum_{i=1}^m (n_i p_i \xi_i) - \frac{1}{2} E(\xi)$.

Proof. (i) The first limit was shown in the proof of Theorem 3.13(ii). The second and third limits in (i) now follow from $\lim_{u \rightarrow -\infty} Y_i(u) = \xi_i$. By L'Hopital's rule we have

$$\lim_{\tau \rightarrow -T_1^+} \frac{a(\tau)}{T_1 + \tau} = \lim_{\tau \rightarrow -T_1^+} \frac{da(\tau)}{d\tau} = \lim_{\tau \rightarrow -T_1^+} \sum_{i=1}^m n_i q_i^2 Y_i(u(\tau))^2 = E(\xi).$$

This proves the last equality in (i).

(ii) By (i) we have

$$(3.30) \quad \lim_{\tau \rightarrow -T_1^+} \frac{b_i(\tau)}{T_1 + \tau} = \lim_{\tau \rightarrow -T_1^+} \left(\frac{b_i(\tau)}{a(\tau)} \cdot \frac{a(\tau)}{T_1 + \tau} \right) = \xi_i^{-1} E(\xi).$$

From this and the proof of Theorem 3.9(i), we get the desired type I curvature estimate as $\tau \rightarrow -T_1^+$.

(iii) The convergence of the rescaled metric follows from (3.30). That the limit is an Einstein metric can be deduced either directly from Ricci curvature formula in Remark 3.10(i) or by appealing to [WZ90, (1.5), (1.6)]. The scalar curvature formula follows from the formula in Remark 3.10(i). \square

Remark 3.19. The topological properties of the circle bundles P_Q we are considering were first investigated in [WZ90]. For dimension 7 a more detailed study of the topological versus differential topological properties was then made by Kreck and Stolz [KS88] at the suggestion of Wang and Ziller. For the convenience of readers not familiar with these works, we summarize below some conclusions which are relevant to the present article.

(i) The manifolds P_Q are simply connected if the q_i are pairwise relatively prime. Let us assume this holds in all of the following. Then for each fixed odd dimension ≥ 7 , there are (countably) infinitely many homotopy types among the P_Q as well as infinitely many homeomorphic types within a fixed integral cohomology ring structure. Such diverse topological behaviour already occurs when the base consists of two KE Fano factors which are complex projective spaces of complex dimension > 1 . Together with the existence results in this section, we obtain type I ancient solutions of the Ricci flow with positive Ricci curvature on all these topologically diverse simply connected manifolds.

(ii) In the case of dimension 7, the work of Kreck and Stolz shows that there is a circle bundle P_Q over $\mathbb{CP}^1 \times \mathbb{CP}^2$ such that its connected sum with any of the 27 exotic 7-spheres is again a circle bundle of the form $P_{\tilde{Q}}$ over $\mathbb{CP}^1 \times \mathbb{CP}^2$. Furthermore, these

bundles are pairwise homeomorphic but not diffeomorphic. Our existence theorem gives type I ancient flows on all these manifolds. We certainly believe that this type of phenomenon holds in higher odd dimensions as well, but at the moment the differential classification in this generality appears to be out of reach.

(iii) Note that the rescaled ancient solutions $g_{a,\bar{b}}(\tau)/(\tau + T_1)$ described in Theorems 3.13(ii) and 3.18 connect the homothety classes of Einstein metrics on P_Q corresponding to ξ (at $\tau = -T_1$) to the homothety class of the product Einstein metric (at $\tau = \infty$) on the lower-dimensional manifold $M_1 \times \cdots \times M_m$.

Now for suitable choices of base manifolds and Q the manifolds P_Q actually become diffeomorphic (see Proposition 2.3 in [WZ90]). The simplest and lowest dimension case occurs when $M = \mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2$. As long as the q_i are nonzero and pairwise relatively prime, the P_Q are all diffeomorphic to $S^2 \times S^3$. However, the Einstein metrics corresponding to ξ_Q actually belong to different path components of the moduli space of Einstein metrics on $S^2 \times S^3$ (since the Einstein constants tend to zero if the volumes are fixed). In such situations, our ancient solutions provide a path in the space of metrics which satisfies a geometric PDE linking these non-isometric Einstein metrics with an Einstein metric on the base space.

(iv) Note also that the Einstein metrics on those circle bundles P_Q for which the ratios $p_i/(-q_i)$ are positive and independent of i are actually Sasakian-Einstein. (Indeed, the corresponding complex line bundles admit a complete asymptotically conical Calabi-Yau metric [WW98].) Therefore, in this special situation one obtains examples of κ -non-collapsed ancient solutions whose backwards limits are Sasakian-Einstein.

3.5. κ -noncollapsed ancient solutions on circle bundles when $m = 2$ and 3. In this subsection we consider the ancient solutions from Theorem 3.6(ii) for the special cases when $m = 2$ and $m = 3$. We will show that $\lim_{u \rightarrow -\infty} Y(u) = v_\theta$ and discuss their geometric properties as $u \rightarrow -\infty$.

First we consider the $m = 2$ case with initial condition $Y(0) \in \Omega_k$ for $k = 1, 2$. Then equation (3.6b) becomes

$$(3.31) \quad \frac{dY_1}{du} = -Y_1 F_1(Y), \quad \frac{dY_2}{du} = -Y_2 F_2(Y),$$

where

$$\begin{aligned} F_i(Y) &= 2p_i Y_i - q_i^2 Y_i^2 - E(Y), \quad i = 1, 2, \text{ and} \\ E(Y) &= n_1 q_1^2 Y_1^2 + n_2 q_2^2 Y_2^2. \end{aligned}$$

Note that Ω_k contains three zeros of vector field $(-Y_1 F_1(Y), -Y_2 F_2(Y))$: 0 , ξ and v_k , where $v_1 = (\frac{2p_1}{(n_1+1)q_1^2}, 0)$ and $v_2 = (0, \frac{2p_2}{(n_2+1)q_2^2})$.

By the linear analysis at ξ given in Lemma 3.3 and Remark 3.14, we conclude that the two branches of the (local) stable curve of $(-Y_1 F_1(Y), -Y_2 F_2(Y))$ at ξ lie respectively inside $\text{Int}(\Omega_1)$ and $\text{Int}(\Omega_2)$ and we will denote them by $\gamma_1(u)$ and $\gamma_2(u)$. We assume that $\gamma_k(0)$ is close to ξ and extend $\gamma_k(u)$ backwards to its maximal time of existence.

Proposition 3.20. (i) $\lim_{u \rightarrow \infty} \gamma_k(u) = \xi$, $k = 1, 2$.
(ii) $\gamma_k(u)$ exists on $(-\infty, \infty)$ and $\lim_{u \rightarrow -\infty} \gamma_k(u) = v_k$, $k = 1, 2$.

Proof. (i) follows from the definition of γ_k .

(ii) We regard $\gamma_k(u)$ as a flow line of the above vector field. By arguments similar to those for Theorem 3.13, one sees that the vector field points outside of Ω_k along the boundary points. Hence going backwards in u the flow lines $\gamma_k(u)$ would stay in Ω_k . Since Ω_k is a compact set, it follows that $\gamma_k(u)$ is defined on all of $(-\infty, \infty)$.

We next consider α_{γ_k} , the α -limit set of flow line γ_k . Since $\gamma_k(u)$ stays in the compact set Ω_k , α_{γ_k} must be non-empty, compact, connected, and flow-invariant (see e.g. [PaM82, Proposition 1.4]). Since solution $\gamma_k(u)$ of (3.31) lies in Ω_k , one easily checks that along $\gamma_k(u)$ one of $\frac{dY_1(u)}{du}$ and $\frac{dY_2(u)}{du}$ is ≥ 0 and the other is ≤ 0 , and equality holds for both if and only if $Y_1 F_1(Y) = Y_2 F_2(Y) = 0$. Hence α -limit set α_{γ_k} is one of $\{0\}, \{v_k\}, \{\xi\}$.

When $Y \in \Omega_1$ is very close to 0, the quantity $\frac{Y_2}{Y_1}$ is very small, hence

$$\begin{aligned} \frac{dE(Y(u))}{du} &= -2n_1 q_1^2 Y_1^2 F_1(Y) - 2n_2 q_2^2 Y_2^2 F_2(Y) \\ &\sim -2n_1 q_1^2 Y_1^2 \cdot (2p_1 Y_1) - 2n_2 q_2^2 Y_2^2 \cdot (-n_1 q_1^2 Y_1^2) < 0, \end{aligned}$$

where for two quantities $A \sim B$ means $\frac{A}{B}$ is close to 1. This implies that α_{γ_1} cannot be 0. By symmetry, α_{γ_2} cannot be 0 either.

If $\alpha_{\gamma_k} = \xi$, then $\lim_{u \rightarrow \infty} \gamma_k(u) = \lim_{u \rightarrow -\infty} \gamma_k(u) = \xi$, by the monotonicity of $\bar{\lambda}(\gamma_k(u))$ from Lemma 3.4 we conclude that $\gamma_k(u) = \xi$ for all u . This is a contradiction. Hence α_{γ_k} cannot be ξ and we get (ii). Actually by Lemma 3.3 v_k is a source. \square

Theorem 3.21. *Let $g_{a,b}^k(\tau)$ defined by functions $a_k(\tau), b_{k1}(\tau), b_{k2}(\tau)$ be the solution of the backwards Ricci flow (3.2a) and (3.2b) corresponding to $\gamma_k(u)$ in Proposition 3.20, $k = 1, 2$. Then we have the following estimates and asymptotics.*

(i) *The maximal interval of existence of $a_k(\tau), b_{k1}(\tau), b_{k2}(\tau)$ is $(-T_k, \infty)$ for some $T_k > 0$, and $\lim_{\tau \rightarrow -T_k^+} a_k(\tau) = 0$.*

(ii) *For the ancient solution $g_{a,b}^1(\tau)$ we have*

$$(3.32a) \quad b_{11}(0) + \left(2p_1 - q_1^2 \frac{a(0)}{b_{11}(0)}\right) \tau \leq b_{11}(\tau) \leq b_{11}(0) + \frac{2n_1}{n_1 + 1} p_1 \tau, \quad \tau \leq 0,$$

$$(3.32b) \quad b_{12}(0) + 2p_2 \tau \leq b_{12}(\tau) \leq b_{12}(0) + \left(2p_2 - q_2^2 \frac{a(0)}{b_{12}(0)}\right) \tau, \quad \tau \leq 0.$$

Hence

$$\lim_{\tau \rightarrow -T_1^+} \frac{a_1(\tau)}{T_1 + \tau} = E(v_1), \quad \lim_{\tau \rightarrow -T_1^+} \frac{b_{11}(\tau)}{T_1 + \tau} = \frac{(n_1 + 1)q_1^2}{2p_1} E(v_1), \quad \lim_{\tau \rightarrow -T_1^+} b_{12}(\tau) > 0.$$

Geometrically, as $\tau \rightarrow -T_1^+$, the circle bundle P_Q with two base factors, equipped with the metric $g_{a,b}^1(\tau)$, collapses to a multiple of the Einstein metric on M_2 in Gromov-Hausdorff topology. Similar conclusions can be made for the solution $g_{a,b}^2(\tau)$.

Proof. (i) We have proved in Theorem 3.8 that the solutions exist at least on $[0, \infty)$. An argument similar to that in the proof of Theorem 3.13(ii), gives $\lim_{u \rightarrow -\infty} \tau(u) \doteq -T_k > -\infty$ for the solution $g_{a,b}^k(\tau)$ and $\lim_{\tau \rightarrow -T_k^+} a_k(\tau) = 0$.

(ii) Since $\gamma_1(u)$ is in Ω_1 , we have $\frac{dY_1}{du} \leq 0$ and $\frac{dY_2}{du} \geq 0$ along $\gamma_1(u)$. Hence we have that for all $\tau \leq 0$

$$0 < 2p_1 - q_1^2 \cdot \frac{2p_1}{(n_1+1)q_1^2} \leq \frac{db_{11}}{d\tau} \leq 2p_1 - q_1^2 Y_1(0),$$

$$2p_2 - q_2^2 Y_2(0) \leq \frac{db_{12}}{d\tau} \leq 2p_2,$$

where we have used $Y_1(u) \leq \frac{2p_1}{(n_1+1)q_1^2}$ to get the first inequality above. Integrating these inequalities we get (3.32a) and (3.32b).

The first limit follows from $\lim_{\tau \rightarrow -T_1^+} \frac{da_1}{d\tau} = E(v_1)$. Since $\lim_{\tau \rightarrow -T_1^+} \frac{b_{11}(\tau)}{a_1(\tau)} = \lim_{u \rightarrow -\infty} \frac{1}{Y_1(u)} = \frac{(n_1+1)q_1^2}{2p_1}$, the second limit follows from this and the first limit.

By moving the initial condition $Y(0)$ close to v_1 , we may assume that $\frac{4p_2 Y_2(0)}{E(v_1)} < 1$ and that $|E(Y(u)) - E(v_1)| \leq \frac{1}{2}E(v_1)$ for all $u \leq 0$. We have $\frac{da_1}{d\tau} = E(Y(u)) \geq \frac{1}{2}E(v_1)$. Integrating the inequality over $(-T_1, 0]$ we get $a_1(0) \geq \frac{1}{2}E(v_1)T_1$. From (3.32b) we get

$$b_{12}(\tau) \geq b_{12}(0) - 2p_2 T_1 \geq b_{12}(0) - \frac{4p_2 a_1(0)}{E(v_1)} \geq b_{12}(0) \left(1 - \frac{4p_2 Y_2(0)}{E(v_1)}\right) > 0.$$

This proves the third limit. \square

Now we discuss the curvature properties of the ancient solutions $g_{a,b}^k(\tau)$ for τ close to $-T_k^+$. Note that the curvature properties for τ close to ∞ have been addressed in Theorem 3.11.

Theorem 3.22. *Let $g_{a,b}^k(\tau)$ be the ancient solution of the Ricci flow on a circle bundle P_Q with two base factors corresponding to $\gamma_k(u)$ in Proposition 3.20, $k = 1, 2$. Then (i) $g_{a,b}^k(\tau)$ develops a singularity of type I as $\tau \rightarrow -T_k^+$. By Theorem 3.21(ii) the type I rescaled metric $(T_k + \tau)^{-1} g_{a,b}^k(\tau)$ converges in the Gromov-Hausdorff topology to the Riemannian product of the Einstein metric*

$$E(v_1) \left(\sigma(\cdot) \otimes \sigma(\cdot) + \frac{(n_1+1)q_1^2}{2p_1} g_1 \right)$$

on the circle bundle over M_1 and the Euclidean metric on \mathbb{R}^{2n_2} ;

(ii) there is a $\kappa > 0$ such that $g_{a,b}^k(\tau)$, $\tau \in (-T_k, 0]$, is κ -noncollapsed at all scales.

Proof. (i) Note that Theorem 3.21(ii) gives the asymptotic behavior of the functions $a_k(\tau)$ and $b_{ki}(\tau)$ as $\tau \rightarrow -T_k^+$. An inspection of the proof of Theorem 3.9(i) gives

$$(T_k + \tau) \cdot \sup_{x \in P_Q} |\text{Rm}_{g_{a,b}^k(\tau)}(x)|_{g_{a,b}(\tau)} \leq C < \infty.$$

The convergence of $(T_k + \tau)^{-1} g_{a,b}^k(\tau)$ is obvious.

Following the definition of torus bundle P_Q at the beginning of §2, the circle bundle on M_1 is the bundle with Euler class $\frac{1}{2\pi} q_1 [\omega_1]$. Note that $F_1(Y_1) = 2p_1 Y_1 - (n_1+1)q_1^2 Y_1^2$ for this circle bundle with $m = 1$. We have $Y_1 = \frac{a}{b_1} = \frac{2p_1}{(n_1+1)q_1^2}$ for metric $\sigma(\cdot) \otimes \sigma(\cdot) + \frac{(n_1+1)q_1^2}{2p_1} g_1$. Since this Y_1 is a solution of $F_1(Y_1) = 0$, this metric is Einstein by Remark 3.1.

(ii) Since the Einstein metric $\sigma(\cdot) \otimes \sigma(\cdot) + \frac{(n_1+1)q_1^2}{2p_1}g_1$ is not flat, (ii) follows from the same argument in the proof of Theorem 3.11(ii). \square

Next we consider one of the κ -noncollapsed ancient solutions from Theorem 3.6(ii) which are defined on circle bundles P_Q with $m = 3$. It follows from Theorem 3.6(i) and Lemma 3.5 that the corresponding flow line $Y(u)$ stays in a bounded set, and hence it exists on $(-\infty, \infty)$. To find the limit of $Y(u)$ as $u \rightarrow -\infty$, let α_Y be the α -limit set of $Y(u)$. Then α_Y must be non-empty, compact, connected, and flow-invariant.

If α_Y contains only fixed points of the flow, then since $m = 3$, α_Y must be one of $0, v_{\{1\}}, v_{\{2\}}, v_{\{3\}}, v_{\{1,2\}}, v_{\{1,3\}}, v_{\{2,3\}}$, or ξ . We can ruled out 0 since 0 is an attractor of the flow by Theorem 3.6(i). We can ruled out ξ by the monotonicity of $\bar{\lambda}$ as in the proof of Proposition 3.20(ii). Hence we conclude that when α_Y contains only fixed points, then $\lim_{u \rightarrow -\infty} Y(u)$ is equal to one of $v_{\{1\}}, v_{\{2\}}, v_{\{3\}}, v_{\{12\}}, v_{\{13\}}, v_{\{23\}}$.

If α_Y contains points other than fixed points, then by a proof analogous to that of Lemma 3.5(iii) we conclude that α_Y is a subset of some coordinate plane. Without loss of generality we may assume that α_Y is a bounded flow line $Y^\infty(u) = (Y_1^\infty(u), Y_2^\infty(u), 0)$ where $Y_1^\infty(u) > 0$ and $Y_2^\infty(u) \geq 0$. By [Hr64, Theorem VII.1.2], $Y^\infty(u)$ is defined on $(-\infty, \infty)$. There are two cases. In the case $Y_2^\infty(u) = 0$ for some u , the flow line is of the form $(Y_1^\infty(u), 0)$ by Lemma 3.5 (i.e., we are actually in the $m = 1$ case) and must flow from v_1 to 0 . However the corresponding $Y^\infty(u)$ cannot be the α -limit set of flow $Y(u)$ since 0 is an attractor of the flow. Therefore, we must be in the second case with $Y_2^\infty(u) > 0$ for all u . We are then reduced to the $m = 2$ situation and the flow line $(Y_1^\infty(u), Y_2^\infty(u))$ must be one of the two flow lines described in Proposition 3.20 which flows from v_k to ξ . But since $(v_k, 0)$ is a repeller of our flow by Lemma 3.3(ii), the corresponding $Y^\infty(u)$ cannot be the α -limit set of flow $Y(u)$. Hence the case of $Y_2^\infty(u) > 0$ is also ruled out, and we have therefore ruled out the possibility that α_Y contains points other than fixed points.

The argument above proves the existence of flow lines $Y(u)$ from one of $v_{\{1\}}, v_{\{2\}}, v_{\{3\}}, v_{\{12\}}, v_{\{13\}}, v_{\{23\}}$ to ξ . Furthermore we can argue as in the proof of Theorem 3.22 and get

Theorem 3.23. *When $m = 3$, each ancient solution $g_{a,\bar{b}}(\tau)$ in Theorem 3.6(ii), which is parametrized by the directions in a circle, develops a type I singularity at some finite time $-T < 0$. The rescaled metrics $(T + \tau)^{-1}g_{a,\bar{b}}(\tau)$ converge to the Riemannian product of some Einstein metric and a Euclidean space as $\tau \rightarrow -T^+$.*

Remark 3.24. For the ancient solutions in Theorem 3.6(ii) when $m = 3$, by dividing up the parametrizing circle, one may speculate that for each $v_{\{k\}}$, $k = 1, 2, 3$, there is a one-parameter family of ancient solutions $Y(u)$ which connect $v_{\{k\}}$ to ξ . Moreover, these families are separated by the three ancient solutions connecting $v_{\{12\}}$, $v_{\{13\}}$, and $v_{\{23\}}$ with the Einstein point ξ . For any $m \geq 3$, under the assumption that $p_i/|q_i|$ is independent of i , we actually can verify a statement similar to the speculation.

Remark 3.25. In [Bo15] and [BLS16] the authors studied properties of the homogeneous Ricci flow and proved some general structure theorems about the existence and behavior of finite time singularities. There is also an unpublished preprint [Buz2] about the homogeneous Ricci flow for compact homogeneous spaces whose isotropy

representation is multiplicity free and satisfies further technical conditions, but otherwise assumes no bound on the number of irreducible summands. Some of what we deduced in Theorems 3.18, 3.22, and 4.5 below about finite-time convergence or collapse at Einstein metrics are special cases of what these authors proved when all of the Fano KE base factors in P_Q are compact homogeneous Kähler manifolds.

4. Ancient solutions of Ricci flow on torus bundles of rank $r > 1$

Let P_Q be a torus bundle of rank r over a product of Fano KE manifolds $M_1 \times \cdots \times M_m$ which satisfies the non-degeneracy assumption in §2. In this section we will consider the $r > 1$ case of the backwards Ricci flow (2.5a) and (2.5b) and establish the existence and asymptotic geometric properties of the ancient solutions $g_{h,\vec{b}}(\tau)$ defined by (2.2). The method of proving existence is a generalization of the method used to prove Theorem 3.6(i). We will use the notation of §2.

Recall that we have fixed a decomposition of the torus $T^r = S^1 \times \cdots \times S^1$ and hence we have a corresponding basis $\{e_\alpha; \alpha = 1, \dots, r\}$ for the Lie algebra \mathfrak{t} of T^r . As usual let $(h^{\alpha\beta}(\tau))_{r \times r}$ denote the inverse of matrix $(h_{\alpha\beta}(\tau))_{r \times r}$. We begin with the following simple lemma.

Lemma 4.1. *The following estimates hold for solutions of (2.5a) and (2.5b):*

- (i) $b_i(\tau) \leq 2p_i\tau + b_i(0)$,
- (ii) $0 \leq h^{\alpha\alpha}(\tau) \leq h^{\alpha\alpha}(0)$ and $|h^{\alpha\beta}(\tau)| \leq C_{\alpha\beta}$ for some constant $C_{\alpha\beta}$,
- (iii) there is a constant $c > 0$ such that matrix $(h_{\alpha\beta}(\tau)) \geq cI_{r \times r}$.

Proof. We first note that since the system (2.5a) and (2.5b) represent, up to a sign, a special case of the Ricci flow, the matrix $(h_{\alpha\beta}(\tau))$ is automatically positive definite as long as the flow exists at $t = -\tau$. Likewise, $b_i(\tau) > 0$.

(i) In (2.5b) we have $\sum_{\tilde{\alpha}, \tilde{\beta}=1}^r q_{\tilde{\alpha}i} q_{\tilde{\beta}i} h_{\tilde{\alpha}\tilde{\beta}}(\tau) \geq 0$ for each i , hence $\frac{d}{d\tau} b_i \leq 2p_i$ and the estimate follows.

(ii) It follows from (2.5a) that

$$\frac{d}{d\tau} h^{\alpha\alpha} = - \sum_{i=1}^m \frac{n_i q_{\alpha i}^2}{b_i^2(\tau)} \leq 0.$$

Hence we get the first estimate in (ii). The second estimate then follows from the fact that matrix $(h^{\alpha\beta}(\tau))$ is positive definite.

(iii) This follows from (ii). □

To analyze the system (2.5a) and (2.5b) further, we will use the dependent variables $h_{\alpha\beta}$ together with new dependent variables

$$(4.1) \quad \hat{Y}_i \doteq \frac{\hat{a}}{b_i}, \quad 1 \leq i \leq m$$

where

$$(4.2) \quad \hat{a} \doteq \sum_{\alpha=1}^r h_{\alpha\alpha} = \text{tr } H,$$

and H denotes the self-adjoint linear operator on \mathbb{R}^r associated to the symmetric tensor h via the background Euclidean metric $\langle \cdot, \cdot \rangle$. We introduce the vector $\hat{Y} =$

$(\hat{Y}_1, \dots, \hat{Y}_m)$ as well as the new independent variable \hat{u} given by

$$(4.3) \quad \hat{u} = \hat{u}(\tau) = \int_0^\tau \frac{1}{\hat{a}(\zeta)} d\zeta.$$

Before transforming the system (2.5a) and (2.5b) to our new variables, it is convenient to introduce in addition the vectors

$$(4.4) \quad Q^{(j)} \doteq \sum_{\alpha=1}^r q_{\alpha j} e_\alpha, \quad 1 \leq j \leq m,$$

where $\{e_1, \dots, e_r\}$ is the standard basis of \mathbb{R}^r from §2, and the vectors

$$(4.5) \quad v_\alpha \doteq \sum_{i=1}^m q_{\alpha i} \frac{\sqrt{n_i}}{b_i} \tilde{e}_i, \quad 1 \leq \alpha \leq r,$$

where $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ is the standard basis of \mathbb{R}^m . The vectors v_α were first introduced in [WZ90] and the matrix

$$V \doteq (V_{\alpha\beta} = \langle v_\alpha, v_\beta \rangle)_{r \times r}$$

is actually positive definite since the matrix Q has rank r by the non-degeneracy assumption.

Notice that the equations (2.5a) can now be written as

$$(4.6) \quad \frac{dH}{d\tau} = HVH,$$

so that

$$(4.7) \quad \frac{dH}{d\hat{u}} = \frac{1}{\hat{a}} (H(\hat{a}^2 V)H).$$

Since $\hat{a}^2 V$ is a quadratic function of the \hat{Y}_i , (4.7) is an equation involving \hat{Y} and H . Moreover, using the definition of the vectors v_α , we obtain

$$(4.8) \quad \text{tr}(HVH) = \sum_{i=1}^m \sum_{\alpha=1}^r \frac{n_i}{b_i^2} ((HQ)_{\alpha i})^2 = \frac{1}{\hat{a}^2} \sum_{i=1}^m \sum_{\alpha=1}^r n_i \hat{Y}_i^2 ((HQ)_{\alpha i})^2.$$

Together with

$$(4.9) \quad \frac{d\hat{a}}{d\tau} = \text{tr}(HVH),$$

we obtain

$$(4.10) \quad \frac{d\hat{a}}{d\hat{u}} = \frac{1}{\hat{a}} \sum_{\alpha=1}^r \sum_{i=1}^m n_i \hat{Y}_i^2 ((HQ)_{\alpha i})^2.$$

Next, noting that

$$(4.11) \quad h(Q^{(i)}, Q^{(i)}) = \sum_{\tilde{\alpha}, \tilde{\beta}=1}^r q_{\tilde{\alpha}i} q_{\tilde{\beta}i} h_{\tilde{\alpha}\tilde{\beta}},$$

the equations (2.5b) can be expressed as

$$\begin{aligned}
\frac{d\hat{Y}_i}{d\hat{u}} &= \hat{a} \frac{d\hat{Y}_i}{d\tau} \\
&= \hat{Y}_i \left(\text{tr}(H V H) + \hat{Y}_i \frac{h(Q^{(i)}, Q^{(i)})}{b_i} - 2p_i \hat{Y}_i \right) \\
(4.12) \quad &= \hat{Y}_i \left(\frac{\text{tr}(H(\hat{a}^2 V) H)}{\hat{a}^2} + \left(\frac{h(Q^{(i)}, Q^{(i)})}{\hat{a}} \right) \hat{Y}_i^2 - 2p_i \hat{Y}_i \right).
\end{aligned}$$

Since the matrix $H(\tau) = (h_{\alpha\beta}(\tau))$ is positive definite, we have

$$(4.13) \quad 0 \leq \frac{h(Q^{(i)}, Q^{(i)})}{\hat{a}} \leq \frac{\|H\| \|Q^{(i)}\|^2}{\hat{a}} \leq \|Q^{(i)}\|^2$$

where $\|A\|$ denotes the usual matrix norm $\sqrt{\text{tr}(A^t A)}$ (as well as the Euclidean norm, by abuse of notation). Also,

$$(4.14) \quad \frac{\sum_{\alpha=1}^r ((HQ)_{\alpha i})^2}{\hat{a}^2} \leq \frac{(\|H\| \|Q^{(i)}\|)^2}{\hat{a}^2} \leq \|Q^{(i)}\|^2.$$

So setting $c_i \doteq \|Q^{(i)}\|^2$ and using (4.8) we obtain the useful inequality

$$(4.15) \quad \frac{d\hat{Y}_i}{d\hat{u}} \leq \hat{Y}_i \left(r \sum_{j=1}^m n_j c_j \hat{Y}_j^2 + c_i \hat{Y}_i^2 - 2p_i \hat{Y}_i \right).$$

Remark 4.2. The constants c_i defined above clearly depend only on the topology of the bundle. Furthermore, under the non-degeneracy assumption of P_Q , these constants are positive.

We next note that

$$\sum_{i=1}^m 2p_i \hat{Y}_i^2 \geq \frac{2}{m} \left(\sum_{i=1}^m \hat{Y}_i \right)^2,$$

and that there is a positive constant c_0 depending only on the torus bundle (with $c_0 \geq c_i, 1 \leq i \leq m$), such that

$$(4.16) \quad \sum_{i=1}^m \hat{Y}_i \left(r \sum_{j=1}^m n_j c_j \hat{Y}_j^2 + c_i \hat{Y}_i^2 \right) \leq c_0 \left(\sum_{i=1}^m \hat{Y}_i \right)^3.$$

We introduce the quantity $\hat{E}(\hat{Y}) \doteq \sum_{i=1}^m \hat{Y}_i$. By (4.15) we have

$$(4.17) \quad \frac{d\hat{E}}{d\hat{u}} \leq c_0 \hat{E}^3 - \frac{2}{m} \hat{E}^2 = \hat{E}^2 \left(c_0 \hat{E} - \frac{2}{m} \right).$$

The following result establishes the existence of ancient solutions.

Theorem 4.3. *Let P_Q be a torus bundle over a product of Fano KE manifolds $M_1 \times \cdots \times M_m$ which satisfies the non-degeneracy assumption in §2. Choose an initial invariant metric $h(0) = (h_{\alpha\beta}(0))$ on torus T^r and positive constants $b_i(0), 1 \leq i \leq m$, which satisfy*

$$\sum_{i=1}^m \frac{\sum_{\alpha=1}^r h_{\alpha\alpha}(0)}{b_i(0)} < \frac{1}{c_0 m},$$

where c_0 is given in (4.16). Let $h_{\alpha\beta}(\tau)$ and $b_i(\tau)$ be the solution of the initial value problem of the backwards Ricci flow (2.5a) and (2.5b). Then

- (i) the corresponding function $\hat{Y}(\hat{u})$ is defined for all $\hat{u} \in [0, \infty)$, $\hat{Y}_i(\hat{u}) \leq \hat{E}(\hat{u}) = \sum_{i=1}^m \hat{Y}_i \leq \frac{m}{\hat{u}}$ for all i , and hence $\lim_{\hat{u} \rightarrow \infty} \hat{Y}(\hat{u}) = 0$;
- (ii) $h_{\alpha\beta}(\tau)$ and $b_i(\tau)$ are defined for all $\tau \in [0, \infty)$, i.e., the corresponding metric $g_{h,b}(\tau)$ gives an ancient solution of the Ricci flow on P_Q ;
- (iii) for each i and $\tau \in [0, \infty)$,

$$\left(2p_i - \frac{1}{m}\right) \tau + b_i(0) \leq b_i(\tau) \leq 2p_i\tau + b_i(0).$$

Proof. We begin with the system (4.7) and (4.12) of $(H(\hat{u}), \hat{Y}(\hat{u}))$ and the initial data which satisfy $\hat{E}(\hat{Y}(0)) < \frac{1}{c_0 m}$. Let $[0, \hat{u}_*)$ be the maximal interval on which the solution exists. Since the above system is real analytic in H and \hat{Y} , it follows that $\hat{Y}_i, 1 \leq i \leq m$, remain positive on $[0, \hat{u}_*)$. Equation (4.10) shows that $\hat{a}(\hat{u})$ is an increasing function, so $\hat{a}(\hat{u})$ is positive on $[0, \hat{u}_*)$. We can recover the independent variable τ by

$$\tau(\hat{u}) = \int_0^{\hat{u}} \hat{a}(\zeta) d\zeta.$$

This enable us to recover the matrix $H(\tau)$ and the functions $b_i(\tau)$ from $(H(\hat{u}), \hat{Y}(\hat{u}))$, which then satisfy the system (2.5a) and (2.5b). This implies that H is positive definite, either by invoking properties of the Ricci flow or by examining the ODE for $\det H$, which shows that $\det H$ is increasing in \hat{u} . We have shown the correspondence between the solution $(H(\hat{u}), \hat{Y}(\hat{u}))$ and solution $H(\tau)$ and $b_i(\tau)$.

(i) and (ii) By our assumption on $\hat{E}(\hat{Y}(0))$, it follows from inequality (4.17) that $\hat{E}(\hat{Y}(\hat{u}))$ is a strictly decreasing function on $[0, \hat{u}_*)$. In particular, we have $\hat{Y}_i(\hat{u}) < \frac{1}{c_0 m}$ on $[0, \hat{u}_*)$ for each i . Let $\tau_* \doteq \lim_{\hat{u} \rightarrow \hat{u}_*} \tau(\hat{u})$. By Lemma 4.1(i) we get

$$(4.18) \quad \hat{a}(0) \leq \hat{a}(\tau) = \sum_{\alpha=1}^r h_{\alpha\alpha}(\tau) \leq \frac{1}{c_0 m} (2p_i\tau + b_i(0)) \quad \text{for each } i \text{ and } \tau \in [0, \tau_*].$$

Fix an index i . By (4.18) and (4.3) we have

$$\hat{u}(\tau) \geq \int_0^\tau \frac{c_0 m}{2p_i \zeta + b_i(0)} d\zeta = \frac{c_0 m}{2p_i} (\ln(2p_i\tau + b_i(0)) - \ln b_i(0)).$$

Hence it follows that \hat{u}_* is finite if and only if τ_* is finite. We need to rule out the possibility that τ_* is finite.

We assume below that τ_* is finite. Since $(h_{\alpha\beta})$ is a positive definite matrix, we conclude from (4.18) that $|h_{\alpha\beta}(\tau)| \leq \frac{1}{c_0 m} (2p_i\tau_* + b_i(0))$ for all α and β . By Lemma 4.1(i) $b_i(\tau)$ is bounded on $[0, \tau_*)$. By the extendibility theory of ODE systems applied to (2.5a) and (2.5b), solution $(H(\tau), b_i(\tau))$ can be continued beyond time τ_* , a contradiction.

The remaining assertions in (i) follow from integrating (4.17). Indeed, applying $\hat{E}(\hat{Y}(\hat{u})) < \hat{E}(\hat{Y}(0)) < \frac{1}{c_0 m}$ leads to the inequality $\frac{d\hat{E}}{d\hat{u}} \leq -\frac{1}{m} \hat{E}^2$, which yields

$$0 < \hat{E}(\hat{u}) \leq \frac{m}{\hat{u} + m(\hat{E}(0))^{-1}} \leq \frac{m}{\hat{u}}.$$

(iii) The upper bound is just Lemma 4.1(i). Applying the bound $\hat{Y}_i < \frac{1}{mc_0}$ to (2.5b) together with (4.13), we have

$$(4.19) \quad \frac{db_i}{d\tau} \geq 2p_i - \frac{c_i}{mc_0} \geq 2p_i - \frac{1}{m},$$

since $c_i \leq c_0$. Integrating this inequality then gives the desired lower bound. \square

Corollary 4.4. *Under the same assumptions as in Theorem 4.3 we have*

- (i) $\hat{a}(\tau) = \sum_{\alpha=1}^r h_{\alpha\alpha}(\tau)$ is a bounded strictly increasing function on $[0, \infty)$;
- (ii) the metrics $h(\tau)$ converge, as $\tau \rightarrow \infty$, to a left-invariant metric h_* on T^r .

Proof. (i) We already know that \hat{a} is an increasing function of τ . Combining (4.10) and (4.14) with Theorem 4.3(i) we obtain

$$\frac{1}{\hat{a}} \frac{d\hat{a}}{d\hat{u}} \leq r \sum_{i=1}^m n_i \frac{m^2}{\hat{u}^2} c_i.$$

Setting $\hat{c} = rm^2 \sum_{i=1}^m n_i c_i$, we see that \hat{a} satisfies the differential inequality

$$(4.20) \quad \frac{1}{\phi} \frac{d\phi}{d\hat{u}} \leq \hat{c} \hat{u}^{-2},$$

whose positive solutions are bounded from above, as can be seen upon integration. Let \hat{a}_* denote the limit of $\hat{a}(\hat{u})$ as \hat{u} tends to ∞ .

(ii) To study the convergence of the metrics $h(\tau)$, let X denote an arbitrary unit vector in \mathbb{R}^r . We shall define $h_*(X, X) \doteq \langle H_*(X), X \rangle$ as $\lim_{\hat{u} \rightarrow \infty} \langle H(\hat{u})(X), X \rangle$ and then extend h_* to a symmetric bilinear form in the usual way by polarization and homothety. To justify this definition, we consider

$$\begin{aligned} \frac{d}{d\hat{u}} \langle H(\hat{u})(X), X \rangle &= \left\langle \frac{dH}{d\hat{u}}(X), X \right\rangle \\ &= \hat{a}(\hat{u}) \langle HVH(X), X \rangle \\ &= \hat{a}(\hat{u}) \langle V(HX), HX \rangle > 0, \end{aligned}$$

since V is positive definite and $\hat{a}(\hat{u}) > 0$. It remains to show that $\langle H(X), X \rangle$ is bounded from above.

Now

$$\begin{aligned} (HVH)_{\alpha\beta} &= \frac{1}{\hat{a}^2} \sum_{\tilde{\alpha}, \tilde{\beta}} \sum_i h_{\alpha\tilde{\alpha}} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} n_i h_{\tilde{\beta}\beta} \hat{Y}_i^2 \\ &\leq \frac{1}{\hat{a}^2} \frac{m^2}{\hat{u}^2} \sum_i (HQ)_{\alpha i} (HQ)_{\beta i} n_i \\ &\leq \frac{m^2}{\hat{u}^2} \sum_i c_i n_i, \end{aligned}$$

where we have used Theorem 4.3(i) and (4.14) in the second and third line of the above computation, respectively. In other words, $\|HVH\| \leq \hat{c} \hat{u}^{-2}$ for some constant \hat{c} . It follows that the function $\phi = \langle H(X), X \rangle$ satisfies

$$\frac{1}{\phi} \frac{d\phi}{d\hat{u}} \leq \frac{\hat{a}(\hat{u})}{\langle H(X), X \rangle} \hat{c} \hat{u}^{-2}.$$

But $\hat{a}(\hat{u})$ is bounded from above by \hat{a}_* while $\langle H(X), X \rangle \geq c > 0$ by Lemma 4.1(iii). So ϕ also satisfies a differential inequality of type (4.20) (with a different constant) and hence $\langle H(X), X \rangle$ is bounded from above. It follows that the metrics $h(\tau)$ converge to a limit left-invariant metric. \square

In order to examine the behavior of the curvature along the ancient solutions in Theorem 4.3, we proceed as in §3.3, using the fact that for each fixed τ our metrics make P_Q into a Riemannian submersion onto the base with totally geodesic fibres. Since the computations are similar we shall be brief and only indicate the necessary changes.

Note first that any left-invariant metric on T^r is automatically right-invariant and has zero curvature. Therefore the components of the curvature tensor that need to be computed are

$$g(R_{X,U}(Y), V), \quad g(R_{U,V}(X), Y), \quad g(R_{X,Y}(Z), W)$$

where X, Y, Z, W denote horizontal basic vectors and U, V denote vertical vectors. As before the components $g(R_{X,Y}(Z), U)$ are zero.

We need to choose an orthonormal basis $\{U_1, \dots, U_r\}$ of $h(\tau)$ for our computations. A convenient choice is to let

$$U_\alpha \doteq \sum_{\beta=1}^r B^{\beta\alpha} e_\beta$$

where B is a square root of H . Since $H^{-1} = (B^{-1})^2$ and

$$\text{tr}(H^{-1}(\tau)) \leq \text{tr}(H^{-1}(0)) \doteq \nu_0^2$$

as a result of the equation $\frac{d}{d\tau} H^{-1} = -V$, it follows that

$$|B^{\alpha\beta}| \leq \|B^{-1}\| \leq \nu_0.$$

We complete $\{U_1, \dots, U_r\}$ to an orthonormal frame for P_Q by adding the adapted horizontal orthonormal basis $\{\tilde{e}_j^{(i)} = \frac{1}{\sqrt{b_i}} e_j^{(i)}, 1 \leq j \leq 2n_i, 1 \leq i \leq m\}$.

We will estimate the curvature tensor of $g_{a,\bar{b}}(\tau)$ using the formulas in [Bes87, Theorem 9.28]. We now consider the terms $g(\nabla_{U_\alpha} A)_X Y, U_\beta)$, which occur in both $g(R_{X,U_\alpha}(Y), U_\beta)$ and $g(R_{U_\alpha,U_\beta}(X), Y)$. By the computation on p. 243 of [WZ90] and the fact that we have toral fibres, it suffices to analyse terms of the form $g(\mathcal{L}_{U_\alpha}(X), \mathcal{L}_{U_\beta}(Y))$ where the skew-adjoint operator \mathcal{L}_U is defined by

$$g(\mathcal{L}(X), Y) = \frac{1}{2} h(\sigma(U), F(X, Y)).$$

(Recall that σ, F are respectively the connection and curvature forms of the bundle.) Because the KE factors in the base are orthogonal with respect to F we may let

$X = \tilde{e}_k^{(i)}$ and $Y = \tilde{e}_\ell^{(i)}$ both of which are tangent to M_i . Then

$$\begin{aligned} g(\mathcal{L}_{U_\alpha}(X), \mathcal{L}_{U_\beta}(Y)) &= \frac{1}{4} \sum_{j=1}^{2n_i} h(U_\alpha, F(X, \tilde{e}_j^{(i)})) h(U_\beta, F(Y, \tilde{e}_j^{(i)})) \\ &= \frac{1}{4b_i} \sum_{\tilde{\alpha}, \tilde{\beta}} \sum_{j=1}^{2n_i} h(U_\alpha, e_{\tilde{\alpha}}) h(U_\beta, e_{\tilde{\beta}}) q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \omega_i(X, e_j^{(i)}) \omega_i(Y, e_j^{(i)}) \\ &= \frac{\delta_{k\ell}}{4b_i^2} \sum_{\tilde{\alpha}, \tilde{\beta}} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} h(U_\alpha, e_{\tilde{\alpha}}) h(U_\beta, e_{\tilde{\beta}}). \end{aligned}$$

But

$$\left| \sum_{\tilde{\alpha}, \tilde{\beta}} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} h(U_\alpha, e_{\tilde{\alpha}}) h(U_\beta, e_{\tilde{\beta}}) \right| \leq (\nu_0)^2 \|H\|^2 r^2 \left(\sum_{\tilde{\alpha}} |q_{\tilde{\alpha}i}| \right)^2,$$

and $\|H\|$ is bounded above by \hat{a}_* , so we conclude that

$$\left| g((\nabla_{U_\alpha} A)_{\tilde{e}_k^{(i)}} \tilde{e}_\ell^{(i)}, U_\beta) \right| \sim O(\tau^{-2}).$$

Similarly, we also have

$$\left| g(A_{\tilde{e}_k^{(i)}} U_\alpha, A_{\tilde{e}_\ell^{(i)}} U_\beta) \right| \sim O(\tau^{-2}) \text{ and } |g(A_X Y, A_Z W)| \sim O(\tau^{-2})$$

for X, Y, Z, W chosen from the above orthonormal basis. Furthermore, as in §3.3, the components of the curvature tensor of the base (with respect to the metric $\sum_i b_i g_i$) decay asymptotically as $O(\tau^{-1})$. Hence the norm of the curvature tensor of our metrics decay like $O(\tau^{-1})$, as in the case of the circle bundles.

Theorem 4.5. *Under the assumptions of Theorem 4.3 we have*

(i) *the ancient solution $g_{h,\vec{b}}(\tau)$ on bundle P_Q is of type I as $\tau \rightarrow \infty$, i.e., there is a constant $C < \infty$ such that for $\tau \geq 0$*

$$\tau \cdot \sup_{x \in P_Q} |\text{Rm}_{g_{h,\vec{b}}(\tau)}(x)|_{g_{h,\vec{b}}(\tau)} \leq C;$$

(ii) *as $\tau \rightarrow \infty$, the rescaled metric tensors $\tau^{-1} g_{h,\vec{b}}(\tau)$ on bundle P_Q collapse (in the Gromov-Hausdorff topology) to the Einstein product metric $2 \sum_i p_i g_i$ on the base $M_1 \times \cdots \times M_m$;*

(iii) *for any $\kappa > 0$, the solution $g_{h,\vec{b}}(\tau)$ is not κ -noncollapsed at all scales.*

(iv) *for each τ , the metric $g_{h,\vec{b}}(\tau)$ has positive Ricci curvature.*

Proof. Part (i) has already been proved in the discussion preceding Theorem 4.5. The proofs of (ii) and (iii) are analogous to those for the circle bundle case except that the volume decay rate for the metrics $\tau^{-1} g_{h,\vec{b}}(\tau)$ is now $\tau^{-r/2}$. See the proof of Theorem 3.7(iii) and Theorem 3.9 (ii), respectively.

To see (iv), from (2.3a) and (4.6) the toral components are given by $\frac{1}{2} H V H$. This is positive definite since V is positive definite by our non-degeneracy assumption.

For the base components given in (2.4), since $\hat{Y}_i < \frac{1}{mc_0}$, we compute

$$\begin{aligned} & \sum_{i=1}^m \left(\frac{p_i}{b_i(\tau)} - \sum_{\tilde{\alpha}, \tilde{\beta}=1}^r \frac{1}{2} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\tilde{\alpha}\tilde{\beta}}(\tau)}{b_i^2(\tau)} \right) b_i(\tau) g_i \\ &= \sum_{i=1}^m \left(p_i - \sum_{\tilde{\alpha}, \tilde{\beta}=1}^r \frac{1}{2} q_{\tilde{\alpha}i} q_{\tilde{\beta}i} \frac{h_{\tilde{\alpha}\tilde{\beta}}(\tau)}{b_i(\tau)} \right) g_i \\ &\geq \sum_{i=1}^m \left(p_i - \frac{1}{2m} \right) g_i > 0, \end{aligned}$$

where we used the proof of (4.19) to get the last inequality. \square

Remark 4.6. The asymptotic behaviors of the Ricci and scalar curvatures are exactly as in the circle bundle case. For the convenience the reader we include the formula for scalar curvature here.

$$R_{g_{h,\tilde{g}}(\tau)} = \frac{1}{\hat{a}} \left(2 \sum_{i=1}^m n_i p_i \hat{Y}_i - \frac{1}{2} \sum_{i=1}^m n_i \frac{h(Q^{(i)}, Q^{(i)})}{\hat{a}} \hat{Y}_i^2 \right).$$

Remark 4.7. As in the circle bundles case, if we take 2-torus bundles over a product of three complex projective spaces of different dimensions, we obtain infinitely many homotopy types in even dimensions starting from dimension 14. If the dimensions are all equal but > 1 , then the non-degenerate 2-torus bundles have the same integral cohomology ring but fall into infinitely many homeomorphism types. More information can be found in [WZ90]. Thus diverse topological properties are also observed among the ancient solutions constructed in this section.

5. APPENDIX: EIGENVALUES OF SUM OF A DIAGONAL MATRIX AND A RANK ONE MATRIX

The following algebraic lemma is used to estimate the eigenvalues of \mathcal{L}_ξ in Lemma 3.3. It may be the case that some experts know this fact.

Lemma 5.1. *Consider the matrix*

$$A = \begin{bmatrix} \epsilon_1 a_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_m a_m \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_m \end{bmatrix}$$

where each $a_i > 0$ and $\epsilon_i > 0$. After a suitable permutation we may assume that $\epsilon_1 a_1 \geq \cdots \geq \epsilon_m a_m$.

(i) If $\epsilon_i a_i$ are all distinct, then the eigenvalues λ_i of A are positive and distinct. Assume $\lambda_1 > \cdots > \lambda_m$, then we have estimates

$$(5.1) \quad \epsilon_i a_i < \lambda_i < \epsilon_{i-1} a_{i-1} \quad \text{for } i = 2, \dots, m,$$

$$(5.2) \quad \min_i \{\epsilon_i a_i\} + \sum_{i=1}^m a_i \leq \lambda_1 \leq \max_i \{\epsilon_i a_i\} + \sum_{i=1}^m a_i.$$

Furthermore, there is an eigenvector corresponding to λ_1 with all positive entries.

(ii) Suppose that we have $\epsilon_1 a_1 = \cdots = \epsilon_{k_1} a_{k_1} \doteq c_1; \cdots; \epsilon_{k_1+\cdots+k_{l-1}+1} a_{k_1+\cdots+k_{l-1}+1} = \cdots = \epsilon_{k_1+\cdots+k_l} a_{k_1+\cdots+k_l} = \epsilon_m a_m \doteq c_l$, where the c_i are distinct values of multiplicity k_i . Let $k_0 = 0$. Then the eigenvalues of A are the following: c_j of multiplicity $k_j - 1$ for $j = 1, \dots, l$, and the eigenvalues of the matrix

$$\tilde{A} \doteq \begin{bmatrix} \tilde{\epsilon}_1 \tilde{a}_1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \tilde{\epsilon}_l \tilde{a}_l \end{bmatrix} + \begin{bmatrix} \tilde{a}_1 & \cdots & \tilde{a}_l \\ \vdots & \vdots & \vdots \\ \tilde{a}_1 & \cdots & \tilde{a}_l \end{bmatrix}$$

where $\tilde{a}_j \doteq \sum_{i=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} a_i$ and $\tilde{\epsilon}_j \doteq c_j / \tilde{a}_j$ for each j . Note that the constants $\tilde{\epsilon}_j \tilde{a}_j$ are all distinct and the eigenvalues of \tilde{A} can be estimated using (i).

Proof. (i) Define $f(\lambda) \doteq \det(A - \lambda I_{m \times m})$. We compute

$$\begin{aligned} f(\epsilon_1 a_1) &= \begin{vmatrix} a_1 & a_2 & \cdots & a_m \\ a_1 & a_2 + \epsilon_2 a_2 - \epsilon_1 a_1 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_m + \epsilon_m a_m - \epsilon_1 a_1 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & \cdots & a_m \\ 0 & \epsilon_2 a_2 - \epsilon_1 a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_m a_m - \epsilon_1 a_1 \end{vmatrix} \\ &= a_1 \prod_{k \neq 1} (\epsilon_k a_k - \epsilon_1 a_1). \end{aligned}$$

Analogously, it follows that for any i we have

$$(-1)^{m-i} f(\epsilon_i a_i) = (-1)^{m-i} a_i \prod_{k \neq i} (\epsilon_k a_k - \epsilon_i a_i) > 0.$$

By the intermediate value theorem, the equation $f(\lambda) = 0$ has m solutions with $\lambda_i \in (\epsilon_i a_i, \epsilon_{i-1} a_{i-1})$ for $i = 2, \dots, m$ and $\lambda_1 > \epsilon_1 a_1$.

To get an upper bound and a better lower bound of λ_1 , we use the Perron-Frobenius theorem. Since all the entries of A are positive, the largest eigenvalue λ_1 is the so-called Perron-Frobenius eigenvalue of A whose corresponding eigenspace is one-dimensional and is spanned by an eigenvector with positive entries. Furthermore, for any other eigenvalue of A the corresponding eigenvector has entries consisting of both signs (see [Gan59, p.64], for example, for all these properties). It is well-known that the Perron-Frobenius eigenvalue is bounded from above and below respectively by the maximum and minimum of row sums (see [Gan59, p.76], for example). This implies the estimate of λ_1 in (i).

(ii) We use the following two observations to find all the solutions of eigenvector equation of A ,

$$\left(\begin{bmatrix} c_1 I_{k_1 \times k_1} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & c_l I_{k_l \times k_l} \end{bmatrix} + \begin{bmatrix} a_1 & \cdots & a_m \\ \vdots & \vdots & \vdots \\ a_1 & \cdots & a_m \end{bmatrix} \right) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

Observation 1. We assume that the column eigenvector is of the form

$$(0, \dots, 0, u_{k_1+\dots+k_{j-1}+1}, \dots, u_{k_1+\dots+k_{j-1}+k_j}, 0, \dots, 0)^T,$$

where $j = 1, \dots, l$. Without loss of generality we only work out the case when $j = 1$. Let U_1 be the vector $(u_1, \dots, u_{k_1})^T$, then the eigenvector equation becomes

$$\begin{bmatrix} c_1 U_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{k_1} a_i u_i \\ \sum_{i=1}^{k_1} a_i u_i \\ \vdots \\ \sum_{i=1}^{k_1} a_i u_i \end{bmatrix} = \begin{bmatrix} \lambda U_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence the solutions are: eigenvalue $\lambda = c_1$ which is of multiplicity $k_1 - 1$ and the corresponding eigenspace is defined by the equation $\sum_{i=1}^{k_1} a_i u_i = 0$.

Observation 2. We assume that the eigenvector satisfies condition $u_1 = \dots = u_{k_1} \doteq w_1; \dots; u_{k_1+\dots+k_{l-1}+1} = \dots = u_{k_1+\dots+k_l} = u_m \doteq w_l$. If we let E_j denote the vector $(1, \dots, 1)^T$ in \mathbb{R}^{k_j} , then the eigenvector equation becomes

$$\begin{bmatrix} c_1 w_1 E_1 \\ c_2 w_2 E_2 \\ \vdots \\ c_l w_l E_l \end{bmatrix} + \begin{bmatrix} (\sum_{j=1}^l (\sum_{i=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} a_i) w_j) E_1 \\ (\sum_{j=1}^l (\sum_{i=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} a_i) w_j) E_2 \\ \vdots \\ (\sum_{j=1}^l (\sum_{i=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} a_i) w_j) E_l \end{bmatrix} = \begin{bmatrix} \lambda w_1 E_1 \\ \lambda w_2 E_2 \\ \vdots \\ \lambda w_l E_l \end{bmatrix}.$$

The above equation is equivalent to the following eigenvector equation for λ and vector $(w_1, \dots, w_l)^T$,

$$\left(\begin{bmatrix} \tilde{\epsilon}_1 \tilde{a}_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \tilde{\epsilon}_l \tilde{a}_l \end{bmatrix} + \begin{bmatrix} \tilde{a}_1 & \dots & \tilde{a}_l \\ \vdots & \vdots & \vdots \\ \tilde{a}_1 & \dots & \tilde{a}_l \end{bmatrix} \right) \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

where $\tilde{\epsilon}_j$ and \tilde{a}_j are defined in the statement of our lemma. We may apply part (i) to get l positive distinct eigenvalues for this eigenvalue problem and hence we get l positive distinct eigenvalues of A which are different from c_1, \dots, c_l .

Combining the two observations together we get m linearly independent eigenvectors for A since $(k_1 - 1) + \dots + (k_l - 1) + l = m$. \square

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